Odd Graceful Labeling of the Revised Friendship Graphs

E. M. Badr
Department of scientific computing,
Faculty of Computers & Information,
Benha University, Benha, Egypt,

ABSTRACT
The aim of this paper is to present some odd graceful graphs. In particular, we show that the revised friendship graphs \( F(kC_4), F(kC_6), F(kC_{12}), F(kC_{16}) \) and \( F(kC_{20}) \) are odd graceful where \( k \) is any positive integer. Finally, we introduce a new conjecture. The revised friendship graph \( F(kC_4) \) is odd graceful where \( k \) is any positive integer and \( n = 0 \) (mod 4).

Keywords
Graph Theory, odd graceful labeling, friendship graphs.

1. INTRODUCTION
A graph \( G \) of size \( q \) is odd-graceful, if there is an injection \( \phi \) from \( V(G) \) to \( \{0, 1, 2, \ldots, 2q-1\} \) such that, when each edge \( xy \) is assigned the label or weight \( | \phi (x) - \phi (y) | \), the resulting edge labels are \( \{1, 3, 5, \ldots, 2q-1\} \). This definition was introduced in 1991 by Gnanajothi [1] who proved that the class of odd graceful graphs lies between the class of graphs with \( \alpha \)-labelings and the class of bipartite graphs. Gnanajothi [1] proved that every cycle \( C_n \) is odd graceful if \( n \) is even. It is known that the graphs which contain odd cycles are not odd graceful. Badr [2] used the subdivision notation for odd cycle in order to prove that the subdivision of linear triangular snakes is odd graceful. Badr et al [3] proved that the subdivision of ladders \( 3(L_n) \) is odd graceful.

Rosa [4] proved that the cycle \( C_n \) is graceful if and only if \( n = 0 \) or 3 (mod 4). Solairaju and Muruganantham [5] proved that the revised friendship graphs \( F(kC_4), F(kC_6), F(2kC_4) \) are all even vertex graceful, where \( k \) is any positive integer.

Definition 1.1: A revised friendship graph \( F(kC_n), n \geq 3 \) is defined as a connected graph containing \( k \) copies of \( C_4 \) with a vertex in common.

Example 1.2:

In this paper, we show that the revised friendship graphs \( F(kC_4), F(kC_6), F(kC_{12}), F(kC_{16}) \) and \( F(kC_{20}) \) are odd graceful where \( k \) is any positive integer. Finally, we introduce a new conjecture. The revised friendship graph \( F(kC_4) \) is odd graceful where \( k \) is any positive integer and \( n = 0 \) (mod 4).

2. THE MAIN RESULTS

Theorem 2.1: The revised friendship graph \( F(kC_4) \) is odd graceful, where \( k \) is any positive integer.

Proof:
Let \( G = F(kC_4) \) has \( q \) edges and \( p \) vertices. The graph \( G \) consists of the vertices \( u, u_1, u_2, \ldots, u_{2k} \) and \( v_1, v_2, \ldots, v_{2k} \), where the graph \( G \) consisting \( k \) copies of \( C_4 \) with a vertex \( u \) in common, such that \( u_i \) is put between \( u \) and \( v_i \) where \( i = 1, 2, 3, \ldots, 2k \) and \( f = 1, 2, 3, \ldots, k \). The graph \( G \) has \( q = 4k \) and \( p = 3k + 1 \), as shown in Figure 2.

Figure 2: the revised friendship graph \( F(kC_4) \)

Define \( \phi : V(G) \rightarrow \{0, 1, 2 \ldots, 2q-1\} \) as following:

\[
\begin{align*}
\phi (u) &= 0 \\
\phi (u_i) &= 2q - 2i + 1, \quad i = 1, 2, \ldots, 2k \\
\phi (v_i) &= 2q - 8i + 4, \quad i = 1, 2, \ldots, k \\
\end{align*}
\]

a) \( \max_{v \in V(G)} \phi (v) = \max \{0, \max_{i=1}^{2k} (2q - 2i + 1), \max_{i=1}^{2k} (2q - 8i + 4) \} \)

\( = 2q - 1 \), the maximum value of all odds

Hence, \( \phi (v) \in \{0, 1, 2 \ldots 2q - 1\} \)

b) Clearly, The function \( \phi \) is one-to-one mapping from the vertex set \( G \) to the set \( \{0, 1, 2 \ldots 2q - 1\} \)

c) It remains to show that the labels of the edges of \( G \) are all the odd integers of the interval \( [1, 2q-1] \). The range of \( | \phi (u_i) - \phi (u) | \) is \( \{2q - 2i + 1 : i = 1, 2 \ldots 2k \} \)

Figure 1: the revised friendship graphs \( F(4C_4) & F(2C_4) \)
\[
\{2q - 1, 2q - 3 \ldots 2q - 4k + 1\}
\]

The range of \[\phi(v_i) - \phi(u_{2i+1})\] is \[\{4i - 1 : i = 1, 2 \ldots k\}\]

\[
= \{3, 7 \ldots 4k - 1\}
\]

The range of \[\phi(v_i) - \phi(u_2)\] is \[\{4i - 3 : i = 1, 2 \ldots k\}\]

\[
= \{1, 5 \ldots 4k - 3\}
\]

Hence, \[|\phi(u) - \phi(v)| : u, v \in E(G)\] is \{1, 3, 5 \ldots 2q - 1\}

So the revised friendship graph \(F(kC_6)\) is odd graceful.

**Example 2.2**

![Figure 3: the odd graceful labeling of the revised friendship graph \(F(5C_6)\).](image)

**Theorem 2.3**: The revised friendship graph \(F(kC_6)\) is odd graceful, where \(k\) is any positive integer.

**Proof:**

Let \(G = F(kC_6)\) has \(q\) edges and \(p\) vertices. The graph \(G\) consists of the vertices \(u_i, u_{12i+1}, \ldots u_{2k}\) and \(v_1, v_2, \ldots v_q\), where the graph \(G\) consists of \(c_8\) with a vertex \(u\) in common, such that the vertex \(u\) is the common vertex, \(x_i\) is put between \(u\) and \(v\), \(u_i\) is put between \(u\) and \(x_i\), such that \(i = 1, 2, 3, 2k, j = 1, 2, 3 \ldots k\). The graph \(G\) has \(q = 6k\) and \(p = 5k + 1\), as shown in Figure 4.

![Figure 4: the revised friendship graph \(F(kC_6)\)](image)

**Define \(\phi: V(G) \rightarrow \{0, 1, 2 \ldots 2q-1\}\)** as following:

\[
\phi(u) = 0
\]

\[
\phi(u_i) = 2q - 2i + 1, i = 1, 2, 3 \ldots 2k
\]

\[
\phi(x_i) = (4/3)q - 4i + 2, i = 1, 2, 3 \ldots 2k
\]

\[
\phi(v_i) = (2/3)q - 4i + 3, i - \text{odd (} i = 1, 3, 5 \ldots\)
\]

\[
\phi(v_i) = (2/3)q - 4i + 1, i - \text{even (} i = 2, 4, 6 \ldots\)
\]

\[
\text{Max } \phi(v) = \max (0, \max (2q - 2i + 1), \max ((4/3)q - 4i + 2),\]

\[
\max (\text{max } ((2/3)q - 4i + 3), \max ((2/3)q - 4i + 1)) = 2q - 1, \text{ the maximum value of all odds}
\]

Hence, \(\phi(v) \in \{0, 1, 2 \ldots 2q - 1\}\)

b) Clearly, The function \(\phi\) is one-to-one mapping from the vertex set of \(G\) to the set \(\{0, 1, 2, \ldots 2q - 1\}\)

c) It remains to show that the labels of the edges of \(G\) are all the odd integers of the interval \(\{1, 2q - 1\}\) and that's as following:

The range of \[|\phi(u) - \phi(u)| : u, v \in E(G)\] is \[\{2q - 1, 2q - 3 \ldots 2q - 4k + 1\}\]

The range of \[|\phi(u) - \phi(x)| : u, v \in E(G)\] is \[\{2q - 1, 2q - 3 \ldots 2q - 4k + 1\}\]

The range of \[|\phi(v) - \phi(x)| : u, v \in E(G)\] is \[\{2q - 1, 2q - 3 \ldots 2q - 4k + 1\}\]

Hence, \(|\phi(u) - \phi(v)| : u, v \in E(G)\) is \(\{1, 3, 5 \ldots 2q - 1\}\).

So the revised friendship graph \(F(kC_6)\) is odd graceful.

**Theorem 2.4**: The revised friendship graph \(F(kC_6)\) is odd graceful, where \(k\) is any positive integer.

**Proof:**

Let \(G = F(kC_6)\) has \(q\) edges and \(p\) vertices. The graph \(G\) consists of the vertices \(u_i, u_{12i+1}, \ldots u_{2k}, x_1, x_2, \ldots x_{2k}\) and \(v_1, v_2, \ldots v_q\), where the graph \(G\) consisting \(k\) copies of \(C_8\) with a vertex \(u\) in common, such that the vertex \(u\) is the common vertex, \(x_i\) is put between \(u\) and \(v\), \(u_i\) is put between \(u\) and \(x_i\), such that \(i = 1, 2, 3, 2k, j = 1, 2, 3 \ldots k\). The graph \(G\) has \(q = 8k\) and \(p = 7k + 1\), as shown in the next figure.
Theorem 2.5: The revised friendship graph $F(kC_4)$ is odd graceful, where $k$ is any positive integer.

Proof:
Let $G = F(kC_4)$ has $q$ edges and $p$ vertices. The graph $G$ consists of the vertices $u$, $u_1u_2u_3,...,u_{2q}$, $x_1x_2x_3,...,x_{2q}$, $h_1h_2h_3,...,h_{2q}$, $a_1a_2a_3,...,a_{2q}$, $b_1b_2b_3,...,b_{2q}$ and $v_1v_2v_3,...,v_{2k}$, where the graph $G$ consisting of $k$ copies of $C_{12}$ with a vertex $u$ in common, such that the vertex $u$ is the common vertex, $h_i$ is put between $u$ and $v$, $a_i$ is put between $u$ and $h_i$, $b_i$ is put between $u$ and $a_i$, $x_i$ is put between $u$ and $h_i$, $u_i$ is put between $u$ and $x_i$ where $i = 1, 2, 3...2k$ and $j = 1, 2, 3...k$. The graph $G$ has $q = 12k$ and $p = 11k + 1$, as shown in the next figure.

Define $\phi : V(G) \rightarrow \{0, 1, 2, ... 2q - 1\}$ as following:

\[
\phi (u) = 0
\]

\[
\phi (u_i) = 2q - 2i + 1, \quad i = 1, 2, 3...2k
\]

\[
\phi (x_i) = q - 4i + 2, \quad i = 1, 2, 3...2k
\]

\[
\phi (h_i) = 2q - 2i - 4k + 1, \quad i = 1, 2, 3...2k
\]

\[
\phi (v_i) = 2q - 8i + 4, \quad i = 1, 2, 3...k
\]

a)

Max $\phi (v) = \max \{ 0, \max (2q - 2i + 1), \max (q - 4i + 2), \max (2q - 2i - 4k + 1), \max (2q - 8i + 4) \}$

$= 2q - 1$, the maximum value of all odds

Hence, $\phi (v) \in \{ 0, 1, 2 ... 2q - 1 \}$

b) Clearly, The function $\phi$ is one-to-one mapping from the vertex set of $G$ to the set $\{ 0, 1, 2 ... 2q - 1 \}$

c) It remains to show that the labels of the edges of $G$ are all the odd integers of the interval $[1,2q-1]$ and that's as following:

The range of $| \phi (u_i) - \phi (u) | = | 2q - 2i + 1 |, \quad i = 1, 2, 3...2k$

$= | 2q - 1, 2q - 3 ... 2q - 4k + 1 |$

The range of $| \phi (u_i) - \phi (x_i) | = | q + 2i - 1 |, \quad i = 1, 2, 3...2k$

$= | q + 1, q + 3 ... q + 4k - 1 |$

The range of $| \phi (h_i) - \phi (x_i) | = | q + 2i - 4k + 1 |, \quad i = 1, 2...2k$

$= | q - 4k + 1, q - 4k + 3 ... q - 1 |$

The range of $| \phi (v_i) - \phi (h_{2i}) | = | 4k - 3i + 1 |, \quad i = 1, 2, 3...k$

$= | 4k - 3, 4k - 7 ... 1 |$

The range of $| \phi (v_i) - \phi (h_{2i}) | = | 4k - 4i + 3 |, \quad i = 1, 2, 3...k$

$= | 4k - 1, 4k - 5 ... 3 |$

Hence, $| \phi (u) - \phi (v) | : uv \in E(G) = \{ 1, 3, 5 ... 2q - 1 \}$

So the revised friendship graph $F(kC_4)$ is odd graceful.
The range of \( |\phi(u_1) - \phi(x_i)| \) is \((4/3) q + 2i - 1, i = 1, 2 \ldots 2k\)
\[ \{ (4/3) q + 1, (4/3) q + 3 \ldots (4/3) q + 4k - 1 \} \]

The range of \( |\phi(h_j) - \phi(x_i)| = \{ (4/3) q + 2i - 4k - 1, i = 1, 2 \ldots 2k \} \)
\[ \{ (4/3) q - 4k + 1, (4/3) q - 4k + 3 \ldots (4/3) q - 1 \} \]

The range of \( |\phi(h_j) - \phi(a_i)| = \{ (2/3) q + 2i - 4k - 1, i = 1, 2 \ldots 2k \} \)
\[ \{ (2/3) q - 4k + 1, (2/3) q - 4k + 3 \ldots (2/3) q - 1 \} \]

The range of \( |\phi(a_i) - \phi(b_j)| = \{ q - 2i + 1, i = 1, 2 \ldots 2k \} \)
\[ \{ q - 1, q - 3 \ldots q - 4k + 1 \} \]

The range of \( |\phi(v_j) - \phi(b_{2j-1})| = \{ (1/3) q - 4i + 1, i = 1, 2, 3 \ldots k \} \)
\[ \{ (1/3) q - 3, (1/3) q - 7 \ldots 1 \} \]

The range of \( |\phi(v_j) - \phi(b_{2j})| = \{ (1/3) q - 4i + 3j, i = 1, 2, 3 \ldots k \} \)
\[ \{ (1/3) q - 1, (1/3) q - 5 \ldots 3 \} \]

Hence, \( |\phi(u) - \phi(v)| : u \in E(G) = \{ 1, 3, 5 \ldots 2q - 1 \} \)

So the revised friendship graph \( F(kC_{16}) \) is odd graceful.

**Theorem 2.6:** The revised friendship graph \( F(kC_{16}) \) is odd graceful, where \( k \) is any positive integer.

**Proof:**

Let \( G = F(kC_{16}) \) has \( q \) edges and \( p \) vertices. The graph \( G \) consists of the vertices \( u, v, a_1, v_1, a_2, v_2, \ldots, a_{4k}, v_{4k}, \) with a vertex \( u \) in common, such that the vertex \( u \) is the common vertex, \( d_i \) is put between \( u \) and \( v_j, c_i \) is put between \( u \) and \( d_i \) and \( b_j \) is put between \( u \) and \( c_j \). The graph \( G \) has \( q = 16k \) and \( p = 15k + 1 \), as shown in the next figure.

![Figure 7: the revised friendship graph F(kC_{16})](image)

Define \( \phi: V(G) \rightarrow \{0, 1, 2 \ldots 2q-1\} \) as following:

\[ \phi(u) = 0 \]
\[ \phi(u_i) = 2q - 2i + 1 \quad i = 1, 2, 3 \ldots 2k \]
\[ \phi(h_i) = 2q - 2i - 4k + 1 \quad i = 1, 2, 3 \ldots 2k \]
\[ \phi(a_i) = q - 4i + 2 \quad i = 1, 2, 3 \ldots 2k \]
\[ \phi(b_j) = (1/4) q - 2i + 1 \quad i = 1, 2, 3 \ldots 2k \]
\[ \phi(c_i) = (3/2) q - 4i + 2 \quad i = 1, 2, 3 \ldots 2k \]
\[ \phi(d_i) = q - 2i + 1 \quad i = 1, 2, 3 \ldots 2k \]
\[ \phi(v_i) = (5/4) q - 8i + 4 \quad i = 1, 2, 3 \ldots k \]

\( a) \)

\[ \max_{v \in V(G)} |\phi(v)| = \max \{0, \max (2q - 2i + 1), \max ((1/2) q - 4i + 2), \max (2q - 2i - 4k + 1), \max (q - 4i + 2), \max ((1/4) q - 2i + 1), \max ((3/2) q - 4i + 2), \max (q - 2i + 1), \max ((5/4) q - 8i + 4) \} = \]
\[ = 2q - 1 \]

Hence, \( \phi(v) \in \{0, 1, 2 \ldots 2q - 1\} \)

\( b) \) Clearly, The function \( \phi \) is one-to-one mapping from the vertex set of \( G \) to the set \( \{0, 1, 2 \ldots 2q - 1\} \)

\( c) \) It remains to show that the labels of the edges of \( G \) are all the odd integers of the interval \( [1,2q-1] \) and that's as following:

The range of \( |\phi(u_1) - \phi(u)| = (2q - 2i + 1, i = 1, 2 \ldots 2k) \)
\[ = (2q - 1, 2q - 3 \ldots 2q - 4k + 1) \]

The range of \( |\phi(u_1) - \phi(x_i)| = \{ (3/2) q + 2i - 1, i = 1, 2 \ldots 2k \} \)
\[ = \{ (3/2) q + 1, (3/2) q + 3 \ldots (3/2) q + 4k - 1 \} \]

The range of \( |\phi(h_j) - \phi(c_i)| = \{ (5/4) q - 2i - 4k - 1, i = 1, 2 \ldots 2k \} \)
\[ = \{ (5/4) q - 1, (5/4) q + 3 \ldots (5/4) q - 4k + 1 \} \]

The range of \( |\phi(h_j) - \phi(a_i)| = \{ q + 2i - 4k - 1, i = 1, 2 \ldots 2k \} \)
\[ = \{ q - 4k + 1, q - 4k + 3 \ldots q - 1 \} \]

The range of \( |\phi(b_j) - \phi(b_{2j})| = \{ (3/4) q - 2i - 1, i = 1, 2 \ldots 2k \} \)
\[ = \{ (3/4) q - 1, (3/4) q + 3 \ldots (3/4) q - 4k + 1 \} \]

The range of \( |\phi(c_i) - \phi(b_j)| = \{ (5/4) q - 2i + 1, i = 1, 2 \ldots 2k \} \)
\[ = \{ (5/4) q - 1, (5/4) q + 3 \ldots (5/4) q - 4k + 1 \} \]

The range of \( |\phi(c_i) - \phi(d_i)| = \{ (1/2) q - 2i + 1, i = 1, 2 \ldots 2k \} \)
\[ = \{ (1/2) q - 1, (1/2) q + 3 \ldots (1/2) q - 4k + 1 \} \]

The range of \( |\phi(v_j) - \phi(d_{2j-1})| = \{ (1/4) q - 4i + 1, i = 1, 2 \ldots 3k \} \)
\[ = \{ (1/4) q - 3, (1/4) q - 7 \ldots (1/4) q - 4k + 1 \} \]
The range of $|\phi(v) - \phi(d_{2k})| = \{(1/4)q - 4i + 3, i = 1, 2, 3 \ldots k\}$

$= \{(1/4)q - 1, (1/4)q - 5 \ldots (1/4)q - 4k + 3\}$

Hence, $||\phi(u) - \phi(v)||: u, v \in E(G) = \{1, 3, 5 \ldots 2q - 1\}$.

So the revised friendship graph $F(kC_{2b})$ is odd graceful.

**Theorem 2.7:** The revised friendship graph $F(kC_{2b})$ is odd graceful, where $k$ is any positive integer.

**Proof:**

Let $G = F(kC_{2b})$ has $q$ edges and $p$ vertices. The graph $G$ consists of the vertices $u, u_{1}u_{2}u_{3} \ldots u_{2q}$, $h_{1}h_{2}h_{3} \ldots h_{2k}$, $a_{0}a_{1}a_{2} \ldots a_{2q}$, $b_{0}b_{1}b_{2} \ldots b_{2q}$, $c_{1}c_{2} \ldots c_{2q}$, $d_{1}d_{2}d_{3} \ldots d_{2q}$, $g_{1}g_{2} \ldots g_{2q}$, and $v_{1}v_{2}v_{3} \ldots v_{2k}$, where the graph $G$ consisting $k$ copies of $C_{20}$ with a vertex $u$ in common, such that the vertex $u$ is the common vertex, $g_{j}$ is put between $u$ and $v_{j}$, $z_{j}$ is put between $u$ and $u_{j}$, $d_{j}$ is put between $u$ and $h_{j}$, $h_{j}$ is put between $u$ and $a_{j}$, $a_{j}$ is put between $u$ and $b_{j}$, $b_{j}$ is put between $u$ and $x_{j}$, and $x_{j}$ is put between $u$ and $y_{j}$, where $i = 1, 2, 3 \ldots 2k$ and $j = 1, 2, 3 \ldots k$. The graph $G$ has $q = 20k$ and $p = 19k + 1$, as shown in the next figure.

![Figure 8: the revised friendship graph F(kC_{2b})](image)

Define $\phi: V(G) \rightarrow \{0, 1, 2 \ldots 2q-1\}$ as following:

$\phi(u) = 0$

$\phi(u_{i}) = 2q - 2i + 1, i = 1, 2, 3 \ldots 2k$

$\phi(x_{i}) = (2/5)q - 4i + 2, i = 1, 2, 3 \ldots 2k$

$\phi(h_{i}) = 2q - 2i - 4k + 1, i = 1, 2, 3 \ldots 2k$

$\phi(a_{i}) = (4/5)q - 4i + 2, i = 1, 2, 3 \ldots 2k$

$\phi(b_{i}) = 2q - 2i - 8k + 1, i = 1, 2, 3 \ldots 2k$

$\phi(c_{i}) = (6/5)q - 4i + 2, i = 1, 2, 3 \ldots 2k$

$\phi(d_{i}) = (7/5)q - 2i + 1, i = 1, 2, 3 \ldots 2k$

$\phi(z_{i}) = (8/5)q - 4i + 2, i = 1, 2, 3 \ldots 2k$

$\phi(g_{i}) = (1/5)q - 2i + 1, i = 1, 2, 3 \ldots 2k$

$\phi(v_{i}) = q - 8i + 4, i = 1, 2, 3 \ldots k$

$\phi(v_{2i}) = q - 2i - 1, i = 1, 2, 3 \ldots k$

$\phi(v_{2q+i-1}) = q - 2i + 1, i = 1, 2, 3 \ldots k$

$a)$

Max $\phi(v) = \max(0, \max(2q - 2i + 1), \max((2/5)q - 4i + 2), \max((2q - 2i - 4k + 1), \max((4/5)q - 4i + 2), \max((7/5)q - 2i + 1), \max((8/5)q - 4i + 2, \max((1/5)q - 2i + 1), \max(q - 8i + 4))$

$= 2q - 1$, the maximum value of all odds

Hence, $\phi(v) \in \{0, 1, 2 \ldots 2q - 1\}$

b) Clearly, The function $\phi$ is one-to-one mapping from the vertex set of $G$ to the set $\{0,1,2 \ldots 2q-1\}$

c) It remains to show that the labels of the edges of $G$ are all the odd integers of the interval $[1,2q-1]$ and that's as following:

The range of $|\phi(u_{i}) - \phi(u_{j})| = \{2q - 2i + 1, i = 1, 2 \ldots 2k\}$

$= \{2q - 1, 2q - 3 \ldots 2q - 4k + 1\}$

The range of $|\phi(u_{i}) - \phi(x_{i})| = \{(8/5)q + 2i - 1, i = 1, 2 \ldots 2k\}$

$= \{(8/5)q + 1, (8/5)q + 3 \ldots (8/5)q + 4k - 1\}$

The range of $|\phi(h_{i}) - \phi(x_{i})| = \{(8/5)q + 2i - 4k - 1, i = 1, 2 \ldots 2k\}$

$= \{(8/5)q - 4k + 1, (8/5)q - 4k + 3 \ldots (8/5)q - 1\}$

The range of $|\phi(h_{i}) - \phi(a_{i})| = \{(6/5)q + 2i - 4k - 1, i = 1, 2 \ldots 2k\}$

$= \{(6/5)q - 4k + 1, (6/5)q - 4k + 3 \ldots (6/5)q - 1\}$

The range of $|\phi(a_{i}) - \phi(b_{i})| = \{(6/5)q + 2i - 8k - 1, i = 1, 2 \ldots 2k\}$

$= \{(6/5)q - 8k + 1, (6/5)q - 8k + 3 \ldots (6/5)q - 4k - 1\}$

The range of $|\phi(c_{i}) - \phi(b_{i})| = \{(4/5)q + 2i - 8k - 1, i = 1, 2 \ldots 2k\}$

$= \{(4/5)q - 8k + 1, (4/5)q - 8k + 3 \ldots (4/5)q - 4k - 1\}$

The range of $|\phi(c_{i}) - \phi(d_{i})| = \{(1/5)q + 2i - 1, i = 1, 2 \ldots 2k\}$

$= \{(1/5)q + 1, (1/5)q + 3 \ldots (1/5)q + 4k - 1\}$

The range of $|\phi(d_{i}) - \phi(z_{i})| = \{(1/5)q - 2i + 1, i = 1, 2 \ldots 2k\}$

$= \{(1/5)q - 1, (1/5)q - 3 \ldots 1\}$

The range of $|\phi(z_{i}) - \phi(g_{i})| = \{(7/5)q - 2i + 1, i = 1, 2 \ldots 2k\}$

$= \{(7/5)q - 1, (7/5)q - 3 \ldots (7/5)q - 4k + 1\}$

The range of $|\phi(v_{i}) - \phi(g_{2i})| = \{(4/5)q - 4i + 1, i = 1, 2, 3 \ldots k\}$

$= \{(4/5)q - 3, (4/5)q - 5 \ldots (4/5)q - 4k + 1\}$

The range of $|\phi(v_{i}) - \phi(g_{2i-1})| = \{(4/5)q - 4i + 3, i = 1, 2, 3 \ldots k\}$

$= \{(4/5)q - 1, (4/5)q - 5 \ldots (4/5)q - 4k + 3\}$

Hence, $||\phi(u) - \phi(v)||: u, v \in E(G) = \{1, 3, 5 \ldots 2q - 1\}$.
So the revised friendship graph $F(kC_{20})$ is odd graceful.

Now, we introduce a new conjecture and that's as shown.

**Conjecture 2.8:** The revised friendship graph $F(kC_n)$ is odd graceful where $k$ is any positive integer and $n = 0 \pmod{4}$.

### 3. Conclusion

Graceful and odd gracefulness of a graph are two entirely different concepts. A graph may possess one or both of these or neither. In this paper we introduced the odd graceful labeling of the revised friendship graphs $F(kC_4)$, $F(kC_6)$, $F(kC_{12})$, $F(kC_{16})$ and $F(kC_{20})$ where $k$ is any positive integer. Finally, we introduced a new conjecture "The revised friendship graph $F(kC_n)$ is odd graceful where $k$ is any positive integer and $n = 0 \pmod{4}$.

### 4. References


