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A New Bivariate Odd Generalized Exponential Gompertz Distribution

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Abstract The objective of this study was to present a novel bivariate distribution, which we denoted as the bivariate odd generalized exponential gompertz(BOGE-G) distribution. Other well-known models included in this one include the gompertz, generalized exponential, odd generalized exponential, and odd generalized exponential gompertz distribution. The model introduced here is of Marshall-Olkin type [16]. The marginals of the new bivariate distribution have odd generalized exponential gompertz distribution which proposed by[7]. Closed forms exist for both the joint probability density function and the joint cumulative distribution function. The bivariate moment generating function, marginal moment generating function, conditional distribution, joint reliability function, marginal hazard rate function, joint mean waiting time, and joint reversed hazard rate function are some of the properties of this distribution that have been discussed. The maximum likelihood approach is used to estimate the model parameters. To demonstrate empirically the significance and adaptability of the new model in fitting and evaluating real lifespan data, two sets of real data are studied using the new bivariate distribution. Using the software Mathcad, a simulation research was conducted to evaluate the bias and mean square error (MSE)characteristics of MLE. We found that the bias and MSE decrease as the sample size increases.

Keywords Odd Generalized Exponential, Gompertz Distribution, Joint Probability Density Function, Conditional Probability Density Function, Maximum Likelihood Estimation, Fisher Information Matrix, Simulation

1 Introduction

One of the traditional continuous mathematical models that represent a survival function based on mortality laws is the gompertz distribution. The gompertz, exponential, and generalized exponential distributions can all be used to analyze lifetime data, whereas the exponential only has a constant hazard rate function but the gompertz and generalized exponential distributions only have a monotone hazard rate. The lifespan components of physical systems, the creatures in biological populations, data in reliability and medical investigations, as well as the modeling of human mortality and fitting actuarial tables, all benefited from the use of these distributions. Demographers and actuaries used the gompertz distribution to characterize the distribution of adult lifespans. It served as a model of growth to meet the growth of the tumor. In recent years, computer scientists have begun to use the gompertz distribution to simulate the failure rates of computer programming. This concept can also be applied to network theory and marketing science.

Recently, El-Damcese et al [7] has defined a new four-parameter distribution referred to as odd generalized exponential gompertz (OGE-G) distribution. Tahir et al. [24] introduced a new class of univariate distributions called the odd generalized exponential (OGE) distribution and studied each of the odd generalized exponential Weibull (OGE-W) distribution, the odd generalized exponential Frechet (OGE-Fr) distribution and the odd generalized exponential Normal (OGE-N) distribution. Jafari et al. [13] introduced a new distribution called the Beta-Gompertz (BG) distribution. El-Gohary et al. [8] proposed a new distribution known as the generalized gompertz (GG) distribution, which consists of the E, GE and G distributions.. Eugene et al. [9] introduced a new generalization of the gompertz (G) distribution which results of the application of the Gompertz distribution to the Beta generator. Gupta and kundu [12] compared the generalised exponential (GE) distribution to the well-known gamma or weibull distribution as a possible alternative. Pollard and valkovics [20] were the first to study the gompertz distribution, they both used the incomplete or complete gamma function to determine the moment generating function of the gompertz distribution, and their conclusions are either approximative or left in integral form. Because the hazard rate shapes could be increasing, decreasing, bathtub-shaped, or upside-down bathtub-shaped, this method is adaptable.

The major goal of this paper is to provide a new bivariate odd generalized exponential gompertz (BOGE-G) distribution so that the marginal are (OGE-G) distributions. It is obtained using a technique similar to that of the Marshall and Olkin bivariate exponential model see [16].The paper is organized as follows. Section 2 we introduced that the proposed BOGE-G distribution has four parameters but the scale and location parameters can be easily introduced. Also, the joint cumulative distribution function (CDF), the joint probability density function (PDF), the marginal probability density functions and the conditional probability density functions of (BOGE-G) distribution is derived in section 2. Section 3 introduces the moment generating function. In section 4 some reliability studies are obtained. Section 5 obtains the parameter estimation using MLE. In section 6 data analysis are obtained using two real data sets. In section 7 we introduced the simulation study. Finally, a conclusion for the results is given in section 8.

2 Bivariate Odd Generalized Exponential Gompertz Distribution

In this section we introduce the BOGE-G distribution using a technique similar to that which was used by [16] to define the Marshall and Olkin bivariate exponential distribution. In order to estimate the related joint probability density function, we must start with the joint cumulative function of the suggested bivariate distribution. Finally, this distribution's conditional probability density functions and marginal probability density functions are also derived. Let Z be a random variable has univariate $OGE-G$ distribution with parameters $\alpha, \lambda, c, \beta > 0$ written as $OGE-G(\underline{\Theta_1}, \beta)$, where the vector Θ_1 is defined by $\underline{\Theta_1} = (\alpha, \lambda, c)$, then the corresponding cumulative distribution function is given by

$$F(z;\underline{\Theta_1},\beta)=\left\{1-e^{-\alpha\left[e^{\frac{\lambda}{c}(e^{cz}-1)}-1\right]}\right\}^{\beta}, z\geq 0, \quad (1)$$

where α, λ, c are scale parameters and β is shape parameter.

Then the probability density function takes the following form

$$\begin{aligned} f(z;\underline{\Theta_1},\beta) &= \alpha\beta\lambda e^{cz} e^{\frac{\lambda}{c}(e^{cz}-1)} e^{-\alpha\left[e^{\frac{\lambda}{c}(e^{cz}-1)}-1\right]} \\ &\quad \times\left\{1-e^{-\alpha\left[e^{\frac{\lambda}{c}(e^{cz}-1)}-1\right]}\right\}^{\beta} \quad (2)\end{aligned}$$

2.1 Joint cumulative distribution function

Assume that U_1, U_2 and U_3 are three mutually independent random variables where $U_1 \sim$ odd generalized exponential gompertz $(\underline{\Theta_1}, \beta_1)$, $U_2 \sim$ odd generalized exponential gompertz $(\underline{\Theta_1}, \beta_2)$ and $U_3 \sim$ odd generalized exponential gompertz $(\underline{\Theta_1}, \beta_3)$ distribution. Define $Z_1 = max\{U_1, U_3\}$ and $Z_2 = max\{U_2, U_3\}$. Then we say that the bivariate vector (Z_1, Z_2) has a bivariate odd generalized exponential gompertz distribution with parameters $(\alpha, \beta_1, \beta_2, \beta_3, \lambda, c)$. Denote it by $BOGE-G\ (\underline{\varphi})$. The joint cumulative distribution function of bivariate odd generalized exponential gompertz distribution of the random variables (Z_1, Z_2) can be obtained through the following lemma.

Lemma (1). If $(Z_1, Z_2) \sim BOGE-G(\underline{\varphi})$, then the joint CDF of (Z_1, Z_2) for $z_1 > 0$, $z_2 > 0$, is

Lemma (1). If $(Z_1, Z_2) \sim BOGE-G(\underline{\varphi})$, then the joint CDF of (Z_1, Z_2) for $z_1 > 0$, $z_2 > 0$, is

$$\begin{aligned} F_{BOEG-G}(z_1,z_2) &= \left\{1-e^{-\alpha\left[e^{\zeta_1}-1\right]}\right\}^{\beta_1}\left\{1-e^{-\alpha\left[e^{\zeta_2}-1\right]}\right\}^{\beta_2} \\ &\quad \left\{1-e^{-\alpha\left[e^{\zeta_3}-1\right]}\right\}^{\beta_3}. \quad (3)\end{aligned}$$

Where $z = min\{z_1, z_2\}$, $\zeta_i = \frac{\lambda}{c}(e^{cz_i} - 1)$, $i = 1, 2$ and $\zeta_3 = \frac{\lambda}{c}(e^{cz} - 1)$.

Since the joint CDF of the random variables Z_1 and Z_2 is defined as

$$\begin{aligned} F_{BOGE-G}(z_1,z_2) &= P(Z_1\leq z_1, Z_2\leq z_2) \\ &= P(\max\{U_1,U_3\}\leq z_1, \max\{U_2,U_3\}\leq z_2) \\ &= P(U_1\leq z_1, U_2\leq z_2, U_3\leq \min(z_1,z_2)).\end{aligned}$$

As the random variables $U_i (i = 1, 2, 3)$ are mutually independent, we directly obtain

$$\begin{aligned} F_{BOGE-G}(z_1,z_2) &= P(U_1\leq z_1, U_2\leq z_2, U_3\leq \min(z_1,z_2)) \\ &= F_{OGE-G}(z_1;\beta_1,\underline{\Theta_1})F_{OGE-G}(z_2;\beta_2,\underline{\Theta_1}) \\ &\quad F_{OGE-G}(z;\beta_3,\underline{\Theta_1}). \quad (4)\end{aligned}$$

Substituting from (1) into (4), we obtain (3), which complete the proof of the lemma (1).

2.2 Joint probability density function

The following theorem gives the joint PDF of the Z_1 and Z_2 which is the joint PDF of $BOGE-G(\underline{\varphi})$.

Let

$$\begin{aligned} Q_{z_i} &= 1-e^{-\alpha\left[e^{\zeta_i}-1\right]}, \quad i=1,2 \\ \text{and } Q_z &= 1-e^{-\alpha\left[e^{\zeta_3}-1\right]}.\end{aligned}$$

Theorem(1). If the joint CDF of Z_1 and Z_2 is as in (3), then then the joint PDF of Z_1 and Z_2 takes the form

$$f_{Z_1,Z_2}(z_1,z_2)=\begin{cases} f_1(z_1,z_2) & if \quad 0< z_1<z_2<\infty \\ f_2(z_1,z_2) & if \quad 0<z_2<z_1<\infty \\ f_3(z,z) & if \quad 0<z_1=z_2=z<\infty, \end{cases} \quad (5)$$

Where

$$\begin{aligned} f_1(z_1,z_2) &= f_{OGE-G}(z_1;\beta_1+\beta_3,\underline{\Theta_1})f_{OGE-G}(z_2;\beta_2,\underline{\Theta_1}) \\ &= \alpha^2(\beta_1+\beta_3)\lambda^2 e^{cz_1+\zeta_1-\alpha\left[e^{\zeta_1}-1\right]}\left[Q_{z_1}\right]^{\beta_1+\beta_3-1} \\ &\quad \beta_2 e^{cz_2+\zeta_2-\alpha\left[e^{\zeta_2}-1\right]}\left[Q_{z_2}\right]^{\beta_2-1}, \quad (6)\end{aligned}$$

$$\begin{aligned} f_2(z_1, z_2) &= f_{OGE-G}(z_1; \beta_1, \underline{\Theta_1}) f_{OGE-G}(z_2; \beta_2+\beta_3, \underline{\Theta_1}) \\ &= \alpha^2 \beta_1 \lambda^2 e^{cz_1+\zeta_1-\alpha\left[e^{\zeta_1}-1\right]} \left[Q_{z_1}\right]^{\beta_1-1} \times \\ &\quad (\beta_2+\beta_3) e^{cz_2+\zeta_2-\alpha\left[e^{\zeta_2}-1\right]} \left[Q_{z_2}\right]^{\beta_2+\beta_3-1}, \end{aligned} \quad (7)$$

and

$$\begin{aligned} f_3(z, z) &= \frac{\beta_3}{\beta_1+\beta_2+\beta_3} f_{OGE-G}(z_2; \beta_1+\beta_2+\beta_3, \underline{\Theta_1}) \\ &= \alpha\beta_3\lambda e^{cz+\zeta_3-\alpha\left[e^{\zeta_3}-1\right]} \left[Q_{z_2}\right]^{\beta_1+\beta_2+\beta_3-1}. \end{aligned} \quad (8)$$

The expressions for $f_1(.,.)$ and $f_2(.,.)$ can be obtained simply by taking $\frac{\partial^2 F_{Z_1,Z_2}(z_1,z_2)}{\partial z_1 \partial z_2}$ for $z_1 < z_2$ and $z_1 > z_2$ respectively .
But $f_3(.,.)$ cannot be obtained in the same way. For this reason we use the fact that

$$\int_0^\infty \int_0^{z_2} f_1(z_1, z_2) dz_1 dz_2 + \int_0^\infty \int_0^{z_1} f_2(z_1, z_2) dz_2 dz_1$$

$$+ \int_0^\infty f_3(z, z) dz = 1 \quad (9)$$

$$\begin{aligned} \int_0^\infty \int_0^{z_2} f_1(z_1, z_2) dz_1 dz_2 &= \int_0^\infty \alpha\beta_2\lambda e^{cz_2+\zeta_2-\alpha\left[e^{\zeta_2}-1\right]} \\ &\quad \times \left[Q_{z_2}\right]^{\beta_1+\beta_2+\beta_3-1} dz_2 \\ &= \frac{\beta_2}{\beta_1+\beta_2+\beta_3} \end{aligned} \quad (10)$$

and

$$\begin{aligned} \int_0^\infty \int_0^{z_1} f_2(z_1, z_2) dz_2 dz_1 &= \int_0^\infty \alpha\beta_1\lambda e^{cz_1+\zeta_1-\alpha\left[e^{\zeta_1}-1\right]} \\ &\quad \times \left[Q_{z_1}\right]^{\beta_1+\beta_2+\beta_3-1} dz_2 \\ &= \frac{\beta_1}{\beta_1+\beta_2+\beta_3} \end{aligned} \quad (11)$$

substituting from (10) and (11) into (9) we obtain

$$\begin{aligned} \int_0^\infty f_3(z, z) dz &= \int_0^\infty \alpha\beta_3\lambda e^{cz+\zeta_3-\alpha\left[e^{\zeta_3}-1\right]} \\ &\quad \times \left[Q_z\right]^{\beta_1+\beta_2+\beta_3-1} dz \\ &= \frac{\beta_3}{\beta_1+\beta_2+\beta_3} \end{aligned}$$

thus

$$f_3(z, z) = \alpha\beta_3\lambda e^{cz+\zeta_3-\alpha\left[e^{\zeta_3}-1\right]} \left[Q_z\right]^{\beta_1+\beta_2+\beta_3-1}$$

Figure 1. shows the surface plots of the joint probability density function (PDF) for $\beta_1 = \beta_2 = \beta_3 = 2$, $\lambda = \alpha = 1$ and $c = 1.5, 0.5$ and 2 respectively, it means that this distribution is flexible model that can be utilized to fit and analyze real lifespan data efficiently.

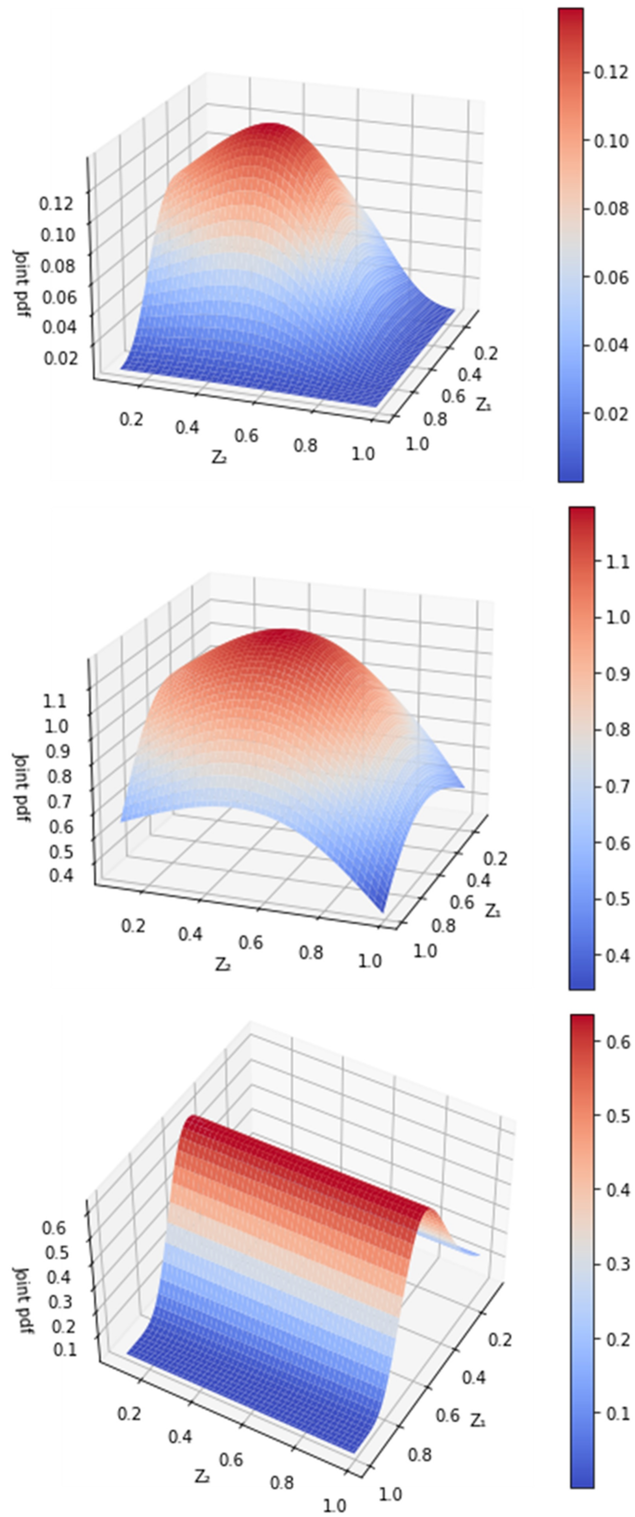

Figure 1. Joint PDF of BOGE-G distribution for β_1= β_2=β_3=2 , $\lambda = \alpha = 1$ and $c = 1.5, 0.5$ and 2 respectively

2.3 Marginal probability density functions

The following theorem gives the marginal probability density functions of Z_1 and Z_2.

Theorem (2). The marginal probability density functions of $Z_i, (i = 1, 2, 3)$ is given by

$$\begin{aligned} f_{Z_i}(z_i) &= f_{OGE-G}(z_i; \beta_i + \beta_2, \underline{\Theta_1}), z_i > 0 \\ &= \alpha(\beta_1 + \beta_3)\lambda e^{cz_i + \frac{\lambda}{c}(e^{cz_i}-1) - \alpha\left[e^{\xi_i}-1\right]} \left[Q_i\right]^{\beta_1+\beta_3-1} \end{aligned} \quad (12)$$

where $i = 1, 2$

The marginal cumulative distribution function of Z_i is

$$\begin{aligned} F(z_i) &= P(Z_i \le z_i) = P(\max\{U_i, U_3\} \le z_i) \\ &= P(U_i \le z_i, U_3 \le z_i). \end{aligned}$$

As the random variables U_i $(i = 1, 2)$ and U_3 are mutually independent,we directly obtain

$$\begin{aligned} F(z_i) &= P(Z_i \le z_i)P(U_3 \le z_i) \\ &= \left[Q_i\right]^{\beta_1}\left[Q_i\right]^{\beta_3} = \left[Q_i\right]^{\beta_1+\beta_3} \\ &= F_{OGE-G}(z_i; \beta_i + \beta_3, \underline{\Theta_1}). \end{aligned} \quad (13)$$

From which we readily derive the pdf of Z_i, $f_{Z_i}(z_i) = \frac{\partial F(z_i)}{\partial z_i}$, as in (12).

where

$$Q_{z_i} = 1 - e^{-\alpha\left[e^{\xi_i}-1\right]}, i = 1, 2$$

Thus, the conditional probability density functions of bivariate distributions are obtained from the joint and marginal distributions $f_{Z_1,Z_2}(Z_1, Z_2), f_{Z_2}(Z_2)$ respectively. Thus, the conditional probability density functions of Z_1 for fixed values of Z_2 is

$$f_{Z_1|Z_2}(Z_1|Z_2) = \frac{f\ (z_1, z_2)}{f\ (z_2)},\ if\ \ f\ (z_2) > 0$$

Theorem (3). The conditional probability density functions of Z_1 given $Z_2 = z_2$ can be expressed as follows

$$f_{Z_1|Z_2}(Z_1|Z_2) = \begin{cases} f^{(1)}_{Z_1|Z_2}(z_1|z_2) & if\ 0 < z_1 < z_2 < \infty \\ f^{(2)}_{Z_1|Z_2}(z_1|z_2) & if\ 0 < z_1 < z_2 < \infty \\ f^{(3)}_{Z_1|Z_2}(z_1|z_2) & if\ 0 < z_1 = z_2 = z < \infty, \end{cases}$$

where

$$f^{(1)}_{Z_1|Z_2}(z_1|z_2) = \frac{\alpha\beta_2(\beta_1+\beta_3)\lambda e^{cz_1+\zeta_1-\alpha\left[e^{\zeta_1}-1\right]}\left[Q_{z_1}\right]^{\beta_1-1}}{(\beta_2+\beta_3)},$$

$$f^{(2)}_{Z_1|Z_2}(z_1|z_2) = \alpha\beta_1\lambda e^{cz_1}e^{\zeta_1}e^{-\alpha\left[e^{\zeta_1}-1\right]}\left[Q_{z_1}\right]^{\beta_1-1},$$

and

$$f^{(3)}_{Z_1|Z_2}(z_1|z_2) = \frac{\beta_3}{\beta_2+\beta_3}\left[Q_{z_1}\right]^{\beta_1-1}.$$

3 Moment Generating Function

In this section, we present the joint moment generating function of (Z_1, Z_2), the marginal moment generating function of Z_1 and used the moment marginal generating function of Z_1 to derive the expectation of Z_1.

3.1 The marginal moment generating function

We drive the marginal moment generating function of Z_1.

Lemma (2). If $Z \sim BOGE-G(\underline{\varphi})$, then the moment generating function of Z_1 is given by

$$\begin{aligned} M_Z(t) &= \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{j}\sum_{l=0}^{\infty}\sum_{m=0}^{l}\binom{\beta_1+\beta_3-1}{i} \\ &\quad \times\frac{U(\beta_1+\beta_3)}{c\,(l-m+1)-t} \end{aligned} \quad (14)$$

where $U = \binom{j}{k}\binom{l}{m}(-1)^{i+j+k+m}\times \frac{\alpha^{j+1}\lambda^{l+1}(i+1)^j(j-k+1)^l}{j!c^l l!}$

Using $f_{Z_1}(z_1)$ in (12) and substituting in

$$M_Z(t) = E(e^{-tz_1}) = \int_0^\infty e^{-tz_1}f_{z_1}(z_1)dz_1,$$

we get

$$\begin{aligned} M_{Z_1}(t) &= \int_0^\infty e^{-tz_1}\alpha\lambda(\beta_1+\beta_3)e^{cz_1+\zeta_1-\alpha\left[e^{\zeta_1}-1\right]} \\ &\quad \times\left\{1-e^{-\alpha\left[e^{\zeta_1}-1\right]}\right\}^{\beta_1+\beta_3-1}dz_1 \end{aligned}$$

by using binomial series expansion of $\left\{1-e^{-\alpha\left[e^{\zeta_1}-1\right]}\right\}^{\beta_1+\beta_3-1}$

$$\begin{aligned} \left\{1-e^{-\alpha\left[e^{\zeta_1}-1\right]}\right\}^{\beta_1+\beta_3-1} &= \sum_{i=0}^{\infty}\binom{\beta_1+\beta_3-1}{i} \\ &\quad \times(-1)^i e^{-\alpha\left[e^{\zeta_1}-1\right]i} \end{aligned}$$

then

$$\begin{aligned} M_{Z_1}(t) &= \alpha\lambda(\beta_1+\beta_3)\sum_{i=0}^{\infty}(-1)^i\binom{\beta_1+\beta_3-1}{i} \\ &\quad \times\int_0^\infty e^{-tz_1+cz_1+\zeta_1-\alpha\left[e^{\zeta_1}-1\right](i+1)}dz_1 \end{aligned} \quad (15)$$

by using taylor series expansion of $e^{-\alpha\left[e^{\zeta_1}-1\right](i+1)}$ we get

$$e^{-\alpha\left[e^{\zeta_1}-1\right](i+1)} = \sum_{j=0}^{\infty}\frac{(-1)^j\alpha^j\,(i+1)^j}{j!}\left(e^{\zeta_1}-1\right)^j$$

therefore,

$$M_{Z_1}(t) = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} \alpha\lambda(\beta_1+\beta_3) \binom{\beta_1+\beta_3-1}{i} (-1)^{i+j} \frac{\alpha^j (i+1)^j}{j!} \times \int_0^{\infty} e^{-tz_1+cz_1+\zeta_1} \left(e^{\zeta_1}-1\right)^j dz_1$$

again using binomial and taylor series expansion, we get

$$M_{Z_1}(t) = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{j}\sum_{l=0}^{\infty}\sum_{m=0}^{l} \binom{\beta_1+\beta_3-1}{i} \times U(\beta_1+\beta_3) \int_0^{\infty} e^{-tz_1+cz_1(l-m+1)} dz_1$$

therefore, we can rewrite (15) as follows:

$$M_{Z_1}(t) = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{j}\sum_{l=0}^{\infty}\sum_{m=0}^{l} \binom{\beta_1+\beta_3-1}{i} \times \frac{U(\beta_1+\beta_3)}{c(l-m+1)-t}$$

3.2 The joint moment generating function

The joint moment generating function of (z_1, z_2) can be derive in next theorem.

Theorem (4). If (z_1, z_2) having $BOGE-G(\underline{\varphi})$ distributions, then the joint moment generating function of $(z_1, \underline{z_2})$ is given by

$$\begin{aligned} M(t_1,t_2) = & \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{j}\sum_{l=0}^{\infty}\sum_{m=0}^{l} \binom{\beta_1+\beta_3-1}{i}\binom{\beta_2-1}{i} \times \frac{\alpha\lambda\beta_2(\beta_1+\beta_3)U}{[-t_1+c(l-m+1)][t_2-c(l-m+1)]} \\ & -\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{j}\sum_{l=0}^{\infty}\sum_{m=0}^{l} \binom{\beta_1+\beta_3-1}{i}\binom{\beta_2-1}{i} \times \frac{\alpha\lambda\beta_2(\beta_1+\beta_3)U}{[-t_1+c(l-m+1)][t_1+t_2-2c(l-m+1)]} \\ & +\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{j}\sum_{l=0}^{\infty}\sum_{m=0}^{l} \binom{\beta_2+\beta_3-1}{i}\binom{\beta_1-1}{i} \times \frac{\alpha\lambda\beta_1(\beta_2+\beta_3)U}{[-t_1+c(l-m+1)][t_2-c(l-m+1)]} \\ & -\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{j}\sum_{l=0}^{\infty}\sum_{m=0}^{l} \binom{\beta_2+\beta_3-1}{i}\binom{\beta_1-1}{i} \times \frac{\alpha\lambda\beta_1(\beta_2+\beta_3)U}{[-t_1+c(l-m+1)][t_1+t_2-2c(l-m+1)]} \\ & +\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{j}\sum_{l=0}^{\infty}\sum_{m=0}^{l} \binom{\beta_1+\beta_2+\beta_3-1}{i} \times \frac{\beta_3 U}{[-(t_1+t_2)+c(l-m+1)]}. \end{aligned}$$

4 Reliability Analysis

The joint reliability function, joint hazard rate, joint mean waiting time, joint reversed (hazard) function, and its marginal function are some of the reliability measures that were introduced in this section.

4.1 Joint reliability function

Assume (Z_1, Z_2) be two dimensional random variables with CDF $F_{Z_1,Z_2}(z_1, z_2)$, and the marginal functions are $F_{Z_1}(z_1)$ and $F_{Z_2}(z_2)$ then, the joint reliability function $R_{Z_1,Z_2}(z_1, z_2)$ is

$$R_{Z_1,Z_2}(z_1,z_2) = 1-F_{Z_1}(z_1)-F_{Z_2}(z_2)+F_{Z_1,Z_2}(z_1,z_2). \quad (16)$$

Assume the random vector (Z_1, Z_2) has the BOGE-G then, the joint reliability function of (Z_1, Z_2) is given by

$$R_{Z_1,Z_2}(z_1,z_2) = \begin{cases} R_1(z_1,z_2) & if\ \ 0<z_1<z_2 \\ R_2(z_1,z_2) & if\ \ 0<z_2<z_1 \\ R_3(z,z) & if\ \ z_1=z_2=z, \end{cases} \quad (17)$$

where

$$R_1(z_1,z_2) = 1-F_{z_1}(z_1)-F_{z_2}(z_2)+F_1(z_1,z_2),$$

it equal to

$$R_1(z_1,z_2) = 1-[Q_{z_1}]^{\beta_1+\beta_3}-[Q_{z_2}]^{\beta_2+\beta_3}+[Q_{z_1}]^{\beta_1+\beta_3}[Q_{z_2}]^{\beta_2}.$$

In addition,

$$R_2(z_1,z_2) = 1-F_{z_1}(z_1)-F_{z_2}(z_2)+F_2(z_1,z_2),$$

it equal to

$$R_2(z_1,z_2) = 1-[Q_{z_1}]^{\beta_1+\beta_3}-[Q_{z_2}]^{\beta_2+\beta_3}+[Q_{z_1}]^{\beta_1}[Q_{z_2}]^{\beta_2+\beta_3}$$

Furthermore

$$R_3(z_1,z_2) = 1-F_{z_1}(z_1)-F_{z_2}(z_2)+F_0(z_2),$$

it has been found that

$$R_3(z,z) = 1-[Q_{z_1}]^{\beta_1+\beta_3}-[Q_{z_2}]^{\beta_2+\beta_3}+[Q_z]^{\beta_1+\beta_2+\beta_3}.$$

4.2 The joint hazard rate function and its marginal functions

Assume (Z_1, Z_2) be two dimensional random variable with PDF $f_{Z_1,Z_2}(z_1, z_2)$, and reliability function $R_{Z_1,Z_2}(z_1, z_2)$. Basu [3] defined the bivariate hazard rate function as

$$h(z_1,z_2) = \frac{f_{Z_1,Z_2}(z_1,z_2)}{R_{Z_1,Z_2}(z_1,z_2)}. \quad (18)$$

Moreover, the bivariate hazard rate function for the random vector (Z_1, Z_2) which has the BOGE-G is

$$h_{Z_1,Z_2}(z_1,z_2) = \begin{cases} h_1(z_1,z_2) & if \quad 0 < z_1 < z_2 \\ h_2(z_1,z_2) & if \quad 0 < z_2 < z_1 \\ h_3(z,z) & if \quad z_1 = z_2 = z\ , \end{cases} \tag{19}$$

where

$$h_1(z_1,z_2) = \frac{f_1(z_1\ ,z_2)}{R_1(z_1,z_2)}.$$

Therefore, we have

$$h_1(z_1,z_2) = \frac{\begin{array}{c}\alpha^2(\beta_1+\beta_3)\lambda^2 e^{cz_1+\zeta_1-\alpha\left[e^{\zeta_1}-1\right]}\left[Q_{z_1}\right]^{\beta_1+\beta_3-1} \\ \beta_2 e^{cz_2+\zeta_2-\alpha\left[e^{\zeta_2}-1\right]}\left[Q_{z_2}\right]^{\beta_2-1}\end{array}}{\begin{array}{c}1-\left[Q_{z_1}\right]^{\beta_1+\beta_3}-\left[Q_{z_2}\right]^{\beta_1+\beta_3} \\ +\left[Q_{z_1}\right]^{\beta_1}\left[Q_{z_2}\right]^{\beta_2+\beta_3\beta_2}\end{array}}$$

$$\begin{aligned} h_2(z_1,z_2) &= \frac{f_2(z_1,z_2)}{R_2(z_1,z_2)} \\ h_2(z_1,z_2) &= \frac{\begin{array}{c}\alpha^2\beta_1\lambda^2 e^{cz_1+\zeta_1-\alpha\left[e^{\zeta_1}-1\right]}\left[Q_{z_1}\right]^{\beta_1-1} \\ (\beta_2+\beta_3)e^{cz_2+\zeta_2-\alpha\left[e^{\zeta_2}-1\right]}\left[Q_{z_2}\right]^{\beta_2+\beta_3-1}\end{array}}{\begin{array}{c}1-\left[Q_{z_1}\right]^{\beta_1+\beta_3}-\left[Q_{z_2}\right]^{\beta_2+\beta_3} \\ +\left[Q_{z_1}\right]^{\beta_1}\left[Q_{z_2}\right]^{\beta_2+\beta_3-1}\end{array}} \end{aligned}$$

and

$$\begin{aligned} h_3(z,z) &= \frac{f_3(z)}{R_3(z,z)} \\ h_3(z,z) &= \frac{\alpha\beta_3\lambda e^{cz+\zeta_3-\alpha\left[e^{\zeta_3}-1\right]}\left[Q_z\right]^{\beta_1+\beta_2+\beta_3-1}}{\begin{array}{c}1-\left[Q_z\right]^{\beta_1+\beta_3}-\left[Q_z\right]^{\beta_2+\beta_3} \\ +\left[Q_z\right]^{\beta_1+\beta_2+\beta_3}\end{array}} \end{aligned}$$

Moreover, the marginal hazard rate functions $h(z_1)$, of the BOGE-G can be obtained from the marginal probability density functions of z_1 and the marginal reliability function of z_1

$$h_{z_1}(z_1) = \frac{\alpha(\beta_1+\beta_3)\lambda e^{cz_1+\zeta_1-\alpha\left[e^{\zeta_1}-1\right]}\left[Q_{z_1}\right]^{\beta_1+\beta_3-1}}{1-\left[Q_{z_1}\right]^{\beta_1+\beta_3}},$$

similarly,

$$h_{z_2}(z_2) = \frac{\alpha(\beta_2+\beta_3)\lambda e^{cz_2+\zeta_2-\alpha\left[e^{\zeta_2}-1\right]}\left[Q_{z_2}\right]^{\beta_2+\beta_3-1}}{1-\left[Q_{z_2}\right]^{\beta_2+\beta_3}}.$$

4.3 The joint mean waiting time and its marginal functions

The waiting time is closely related to another important random variable reversed hazard rate function. In fact, since the reversed hazard function already imposes the requirement of a failure in [0,t], it is useful to describe the amount of time that has passed since the failure in several applications (actuarial science, reliability analysis). One of the most crucial uses of the waiting time is to describe various maintenance strategies to any system. The distribution function can be predicted using the waiting time observations. The joint mean waiting time function $M_w(t_1,t_2)$ is defined as follows

$$M_w(t_1,t_2) = \frac{1}{F(t_1,t_2)}\int_0^{t_1}\int_0^{t_2} F\ (z_1,z_2)\,dz_2dz_1. \tag{20}$$

The following lemma obtains the joint mean waiting time of (z_1, z_2).

Lemma (3). The joint mean waiting time $M_w(t_1,t_2)$ to the random variables Z_1 and Z_2 is

$$M_w(t_1,t_2) = \begin{cases} M_{w_1}(t_1,t_2) & if\ t_1 < t_2 \\ M_{w_2}(t_1,t_2) & if\ \ t_1 > t_2 \\ M_{w_3}(t\ ,t\ ) & if\ t_1 = t_2, \end{cases} \tag{21}$$

where

$$\begin{aligned} M_{w1}(t_1,t_2) = {} & \frac{1}{F(t_1,t_2)}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{j}\sum_{l=0}^{\infty}\sum_{m=0}^{l} \\ & \binom{\beta_1+\beta_3}{i}\binom{j}{k}^2\binom{l}{m}^2 \\ & \binom{\beta_2}{i}(-1)^{2(i+j+k+m)} \\ & \times\frac{(\alpha i)^{2j}\lambda^{2l}(j-k)^{2l}\times\left[e^{c(l-m)t_1}-1\right]\left[e^{c(l-m)t_2}-1\right]}{(j!l!)^2 c^{2(l+1)}(l-m)^2} \end{aligned}$$

$$\begin{aligned} M_{w2}(t_1,t_2) = {} & \frac{1}{F(t_1,t_2)}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{j}\sum_{l=0}^{\infty}\sum_{m=0}^{l} \\ & \binom{\beta_2+\beta_3}{i}\binom{j}{k}^2\binom{l}{m}^2 \\ & \binom{\beta_1}{i}(-1)^{2(i+j+k+m)} \\ & \times\frac{(\alpha i)^{2j}\lambda^{2l}(j-k)^{2l}\times\left[e^{c(l-m)t_1}-1\right]\left[e^{c(l-m)t_2}-1\right]}{(j!l!)^2 c^{2(l+1)}(l-m)^2}, \end{aligned}$$

and

$$\begin{aligned} M_{w3}(t,t) = {} & \frac{1}{F(t,t)}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{j}\sum_{l=0}^{\infty}\sum_{m=0}^{l} \\ & \binom{\beta_1+\beta_2+\beta_3}{i}\binom{j}{k}\binom{l}{m} \\ & \times\frac{(-1)^{i+j+k+m}(\alpha i)^{j}\lambda^{l}\times(j-k)^{l}\left[e^{c(l-m)t}-1\right]}{j!c^{l+1}l!(l-m)}. \end{aligned}$$

4.4 The joint reversed hazard rate function and its marginal functions

Assume (z_1, z_2) be two dimensional random variable with CDF $F_{Z_1,Z_2}(z_1, z_2)$ and pdf $f_{Z_1,Z_2}(z_1, z_2)$ .Then joint reversed hazard rate function is

$$r\ (z_1, z_2) = \frac{f_{Z_1,Z_2}(z_1, z_2)}{F_{Z_1,Z_2}(z_1, z_2)}. \tag{22}$$

Thus, the bivariate reversed hazard rate function for the random vector (z_1, z_2) which has the BOGE-G is

$$r_{Z_1,Z_2}\ (z_1, z_2) = \begin{cases} r_1\ (z_1, z_2) & if \quad 0 < z_1 < z_2 \\ r_2\ (z_1, z_2) & if \quad 0 < z_2 < z_1 \\ r_3\ (z, z) & if \quad z_1 = z_2 = z, \end{cases} \tag{23}$$

Where

$$r_1(z_1, z_2) = \frac{f_1(z_1, z_2)}{F_1(z_1, z_2)}$$

then, we have

$$\begin{aligned} r_1(z_1, z_2) &= \frac{f_1(z_1, z_2)}{F_1(z_1, z_2)} \\ r_1(z_1, z_2) &= \frac{\begin{array}{c}\alpha^2\lambda^2 e^{cz_1+\zeta_1-\alpha\left[e^{\zeta_1}-1\right]}(\beta_1+\beta_3) \\ \times\beta_2 e^{cz_2+\zeta_2-\alpha\left[e^{\zeta_2}-1\right]}\end{array}}{[Q_{z_1}]\,[Q_{z_2}]} \end{aligned}$$

when

$$r_2\ (z_1, z_2) = \frac{f_2(z_1, z_2)}{F_2(z_1, z_2)}$$

therefore, it equal to

$$\begin{aligned} r_2(z_1, z_2) &= \frac{f_2(z_1, z_2)}{F_2(z_1, z_2)} \\ r_2(z_1, z_2) &= \frac{\begin{array}{c}\alpha^2\beta_1\lambda^2 e^{cz_1+\zeta_1-\alpha\left[e^{\zeta_1}-1\right]} \\ \times(\beta_2+\beta_3) e^{cz_2+\zeta_2-\alpha\left[e^{\zeta_2}-1\right]}\end{array}}{[Q_{z_1}]\,[Q_{z_2}]}, \end{aligned}$$

also,

$$r_3\ (z, z_2) = \frac{f_3(z)}{F_3(z)}$$

hence

$$r_3(z, z_2) = \frac{\alpha(\beta_1+\beta_2+\beta_3)\lambda e^{cz+\zeta_3-\alpha\left[e^{\zeta_3}-1\right]}}{[Q_z]}$$

In addition, the marginal reversed hazard rate functions $r_{Z_1}(z_1)$ and $r_{Z_2}(z_2)$ to the BOGE-G are

$$r_{Z_1}(z_1) = \frac{f_{Z_1}(z_1)}{F_{Z_1}(z_1)}$$

it equal to

$$r_{Z_1}(z_1) = \frac{\alpha(\beta_1+\beta_3)\lambda e^{cz_1} e^{\zeta_1} e^{-\alpha\left[e^{\zeta_1}-1\right]}}{[Q_{z_1}]},$$

similarly,

$$r_{Z_2}(z_2) = \frac{\alpha(\beta_2+\beta_3)\lambda e^{cz_2} e^{cz_2+\zeta_2-\alpha\left[e^{\zeta_2}-1\right]}}{[Q_{z_2}]}.$$

5 Maximum Likelihood Estimation

In this section, we use maximum likelihood to estimate the unknown parameters of the BOGE-G distribution. Using the same justification as that given in [10]. we want to estimate the other parameters $(\underline{\varphi})$. Suppose that we have a sample of size n, takes the form $\{(z_1, z_2), ...(z_{1n}, z_{2n})\}$ from BOGE-G distribution.

We employ the notation shown below

$$\begin{aligned} n_1 &= (i; z_{1i} < z_{2i}), n_2 = (i; z_{1i} > z_{2i}) \\ , n_3 &= (i; z_{1i} = z_{2i}) \\ \text{where } n &= n_1 + n_2 + n_3. \end{aligned}$$

Based on the observations, the likelihood function is calculated using the density functions $f_1(z_1, z_2)$, $f_2(z_1, z_2)$ and $f_0(z)$

$$l(\underline{\varphi}) = \prod_{i=1}^{n_1} f_1(z_{1_i}, z_{2i}) \prod_{i=1}^{n_2} f_2(z_{1i}, z_{2i}) \prod_{i=1}^{n_3} f_0(z_i)$$

and let

$$\begin{aligned} H_{ji} &= e^{-\alpha\left[e^{\frac{\lambda}{c}(e^{cz_{ji}}-1)}-1\right]}, \quad j = 1, 2 \\ W_{ji} &= e^{\frac{\lambda}{c}(e^{cz_{ji}}-1)}, \quad j = 1, 2 \end{aligned}$$

The log-likelihood function can be expressed as

$$\begin{aligned} l(\underline{\varphi}) = {} & n_1 \ln\left[\alpha^2\beta_2(\beta_1+\beta_3)\lambda^2\right] + c\sum_{i=1}^{n_1} z_{1i} \\ & + \frac{\lambda}{c}\sum_{i=1}^{n_1}(e^{cz_{1i}}-1) - \alpha\sum_{i=1}^{n_1}(W_{1i}-1) \\ & + (\beta_1+\beta_3-1)\sum_{i=1}^{n_1}\ln(1-H_{1i}) + c\sum_{i=1}^{n_1} z_{2i} \\ & + \frac{\lambda}{c}\sum_{i=1}^{n_1}(e^{cz_{2i}}-1) - \alpha\sum_{i=1}^{n_1}(W_{2i}-1) + (\beta_2-1) \\ & \sum_{i=1}^{n_1}\ln(1-H_{2i}) + n_2\ln\left[\alpha^2\beta_1(\beta_2+\beta_3)\lambda^2\right] \\ & + c\sum_{i=1}^{n_2} z_{1i} + \frac{\lambda}{c}\sum_{i=1}^{n_2}(e^{cz_{1i}}-1) - \alpha\sum_{i=1}^{n_2}(W_{1i}-1) \\ & + (\beta_2+\beta_3-1)\sum_{i=1}^{n_2}\ln(1-H_{1i}) + c\sum_{i=1}^{n_2} z_{2i} \\ & + \frac{\lambda}{c}\sum_{i=1}^{n_2}(e^{cz_{2i}}-1) - \alpha\sum_{i=1}^{n_2}(W_{2i}-1) + (\beta_1-1) \\ & \sum_{i=1}^{n_2}\ln(1-H_{2i}) + n_3\ln[\alpha\beta_3\lambda] + c\sum_{i=1}^{n_3} z_{1i} \\ & + \frac{\lambda}{c}\sum_{i=1}^{n_3}(e^{cz_{1i}}-1) - \alpha\sum_{i=1}^{n3}(W_{1i}-1) \\ & + (\beta_1+\beta_2+\beta_3-1)\sum_{i=1}^{n_3}\ln(1-H_{1i}) \end{aligned}$$

We obtain the likelihood equation by computing the first partial derivatives of the above equation with respect to $\beta_1, \beta_2, \beta_3, \alpha, \lambda$ and c, and setting the results equal zeros as

$$
\begin{aligned}
\frac{\partial l}{\partial \beta_1} &= \frac{n_1}{(\beta_1+\beta_3)} + \sum_{i=1}^{n_1} \ln(1-H_{1i}) + \frac{n_2}{\beta_1} \\
&\quad + \sum_{i=1}^{n_2} \ln(1-H_{2i}) + \sum_{i=1}^{n_3} \ln(1-H_{1i}) = 0
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial l}{\partial \beta_2} &= \frac{n_1}{\beta_2} + \sum_{i=1}^{n_1} \ln(1-H_{2i}) + \frac{n_2}{(\beta_2+\beta_3)} \\
&\quad + \sum_{i=1}^{n_2} \ln(1-H_{1i}) + \sum_{i=1}^{n_3} \ln(1-H_{1i}) = 0
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial l}{\partial \beta_3} &= \frac{n_1}{(\beta_1+\beta_3)} + \sum_{i=1}^{n_1} \ln(1-H_{1i}) + \frac{n_2}{(\beta_2+\beta_3)} \\
&\quad + \sum_{i=1}^{n_2} \ln(1-H_{2i}) + \frac{n_3}{\beta_3} + \sum_{i=1}^{n_3} \ln(1-H_{1i}) = 0
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial l}{\partial c} &= \sum_{i=1}^{n_1} z_{1i} - \sum_{i=1}^{n_1} \frac{-\lambda}{c^2}(e^{cz_{1i}}-1) + \frac{\lambda}{c} z_{1i} e^{cz_{1i}} \\
&\quad -\alpha \sum_{i=1}^{n_1} \frac{-\lambda}{c^2}(e^{cz_{1i}}-1) + \frac{\lambda}{c} z_{1i} e^{cz_{1i}} W_{1i} \\
&\quad +(\beta_1+\beta_3-1)\alpha \\
&\quad \sum_{i=1}^{n_1} \frac{\frac{-\lambda}{c^2}(e^{cz_{1i}}-1) + \frac{\lambda}{c} z_{1i} e^{cz_{1i}} W_{1i}}{H_{1i}-1} \\
&\quad - \sum_{i=1}^{n_1} \frac{-\lambda}{c^2}(e^{cz_{2i}}-1) + \frac{\lambda}{c} z_{2i} e^{cz_{2i}} - \alpha \\
&\quad \sum_{i=1}^{n_1} \frac{-\lambda}{c^2}(e^{cz_{2i}}-1) + \frac{\lambda}{c} z_{2i} e^{cz_{2i}} W_{2i} + (\beta_2-1)\alpha \\
&\quad \sum_{i=1}^{n_1} \frac{\frac{-\lambda}{c^2}(e^{cz_{2i}}-1) + \frac{\lambda}{c} z_{2i} e^{cz_{2i}} W_{2i}}{H_{2i}-1} + \sum_{i=1}^{n_2} z_{1i} \\
&\quad - \sum_{i=1}^{n_2} \frac{-\lambda}{c^2}(e^{cz_{1i}}-1) + \frac{\lambda}{c} z_{1i} e^{cz_{1i}} \\
&\quad -\alpha \sum_{i=1}^{n_2} \frac{-\lambda}{c^2}(e^{cz_{1i}}-1) + \frac{\lambda}{c} z_{1i} e^{cz_{1i}} W_{1i} \\
&\quad +(\beta_2+\beta_3-1)\alpha \\
&\quad \sum_{i=1}^{n_2} \frac{\frac{-\lambda}{c^2}(e^{cz_{1i}}-1) + \frac{\lambda}{c} z_{1i} e^{cz_{1i}} W_{1i}}{H_{1i}-1} \\
&\quad - \sum_{i=1}^{n_2} \frac{-\lambda}{c^2}(e^{cz_{2i}}-1) + \frac{\lambda}{c} z_{2i} e^{cz_{2i}} \\
&\quad -\alpha \sum_{i=1}^{n_2} \frac{-\lambda}{c^2}(e^{cz_{2i}}-1) + \frac{\lambda}{c} z_{2i} e^{cz_{2i}} W_{2i} + (\beta_1-1)\alpha \\
&\quad \sum_{i=1}^{n_2} \frac{\frac{-\lambda}{c^2}(e^{cz_{2i}}-1) + \frac{\lambda}{c} z_{2i} e^{cz_{2i}} W_{2i}}{H_{2i}-1} \\
&\quad + \sum_{i=1}^{n_3} z_{1i} - \sum_{i=1}^{n_3} \frac{-\lambda}{c^2}(e^{cz_{1i}}-1) + \frac{\lambda}{c} z_{1i} e^{cz_{1i}} - \alpha \\
&\quad \sum_{i=1}^{n_3} \frac{-\lambda}{c^2}(e^{cz_{1i}}-1) + \frac{\lambda}{c} z_{1i} e^{cz_{1i}} W_{1i} \\
&\quad +(\beta_1+\beta_2+\beta_3-1)\alpha \\
&\quad \sum_{i=1}^{n_3} \frac{\frac{-\lambda}{c^2}(e^{cz_{1i}}-1) + \frac{\lambda}{c} z_{1i} e^{cz_{1i}} W_{1i}}{H_{1i}-1} = 0
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial l}{\partial \alpha} &= \frac{2n_1}{\alpha} - \sum_{i=1}^{n_1} (W_{1i}-1) - (\beta_1+\beta_3-1) \\
&\quad \sum_{i=1}^{n_1} \frac{1-W_{1i}}{(H_{1i}-1)} - \sum_{i=1}^{n_1} (W_{2i}-1) \\
&\quad -(\beta_2-1) \sum_{i=1}^{n_1} \frac{1-W_{2i}}{(H_{2i}-1)} + \frac{2n_2}{\alpha} \\
&\quad - \sum_{i=1}^{n_2} (W_{1i}-1) - (\beta_2+\beta_3-1) \\
&\quad \sum_{i=1}^{n_2} \frac{1-W_{1i}}{(H_{1i}-1)} - \sum_{i=1}^{n_2} (W_{2i}-1) \\
&\quad -(\beta_1-1) \sum_{i=1}^{n_2} \frac{1-W_{2i}}{(H_{2i}-1)} + \frac{n_3}{\alpha} \\
&\quad - \sum_{i=1}^{n_3} (W_{1i}-1) - (\beta_1+\beta_2+\beta_3-1 \\
&\quad) \sum_{i=1}^{n_3} \frac{1-W_{1i}}{(H_{1i}-1)} = 0
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial l}{\partial \lambda} &= \frac{2n_1}{\lambda} + \frac{1}{c}\sum_{i=1}^{n_1}(e^{cz_{1i}}-1) \\
&-\frac{\alpha}{c}\sum_{i=1}^{n_1}(e^{cz_{1i}}-1)W_{1i} \\
&+\frac{\alpha(\beta_1+\beta_3-1)}{c}\sum_{i=1}^{n_1}\frac{(e^{cz_{1i}}-1)W_{1i}}{(H_{1i}-1)} \\
&+\frac{1}{c}\sum_{i=1}^{n_1}(e^{cz_{2i}}-1) - \frac{\alpha}{c}\sum_{i=1}^{n_1}(e^{cz_{2i}}-1)W_{2i} \\
&+\frac{\alpha(\beta_2-1)}{c}\sum_{i=1}^{n_1}\frac{(e^{cz_{2i}}-1)W_{2i}}{(H_{2i}-1)} \\
&+\frac{2n_2}{\lambda} + \frac{1}{c}\sum_{i=1}^{n_2}(e^{cz_{1i}}-1) - \frac{\alpha}{c} \\
&\sum_{i=1}^{n_2}(e^{cz_{1i}}-1)W_{1i} + \frac{\alpha(\beta_2+\beta_3-1)}{c} \\
&\sum_{i=1}^{n_2}\frac{(e^{cz_{1i}}-1)W_{1i}}{(H_{1i}-1)} + \frac{1}{c}\sum_{i=1}^{n_2}(e^{cz_{2i}}-1) \\
&-\frac{\alpha}{c}\sum_{i=1}^{n_2}(e^{cz_{2i}}-1)W_{2i} + \frac{\alpha(\beta_1-1)}{c} \\
&\sum_{i=1}^{n_2}\frac{(e^{cz_{2i}}-1)W_{2i}}{(H_{2i}-1)} + \frac{n_3}{\lambda} + \frac{1}{c}\sum_{i=1}^{n_3}(e^{cz_{1i}}-1) \\
&-\frac{\alpha}{c}\sum_{i=1}^{n_3}(e^{cz_{1i}}-1)W_{1i} + \frac{\alpha(\beta_1+\beta_2+\beta_3-1)}{c} \\
&\sum_{i=1}^{n_3}\frac{(e^{cz_{1i}}-1)W_{1i}}{(H_{1i}-1)} = 0
\end{aligned}
$$

To get the MLEs of the parameters ($\underset{\sim}{\varphi}$), we have to solve the above system of six non-linear equations with respect to $\beta_1, \beta_2, \beta_3, \alpha, \lambda$ and c. Since it is difficult to solve the above equations, numerical methods must be used to obtain the MLEs, the second derivatives for the BOGE-G distribution are provided in the appendix.

Using the variance-covariance matrix, we can obtain the $(1-\delta)100\%$ confidence intervals for the parameters $\beta_1, \beta_2, \beta_3, \alpha, \lambda$ and c as shown in the following forms.

$$\hat{\beta_i} \pm Z_{\frac{\delta}{2}}\sqrt{Var\left(\hat{\beta_i}\right)},$$

where $Z_{\frac{\delta}{2}}$ is the upper $\left(\frac{\delta}{2}\right)$ th percentile of the standard normal distribution.

6 Data Analysis

By comparing the BOGE-G distribution with other well-known distributions like the bivariate exponentiated generalized weibull gompertz (BEGWG) distribution and the bivariate exponentiated modified weibull (BEMW) distribution, we demonstrate the empirical significance of the BOGE-G distribution. For more information, see [5, 23]. The fitted distributions are compared using a number of criteria, such as the maximum log-likelihood (L), Akaike information criterion (AIC), correct Akaike information criterion (CAIC), Bayesian information criterion (BIC), Hannan-Quinn information criterion (HQIC) test, Bayesian estimation (BSE), and confidence interval, to determine which distribution best fits the data.

6.1 The First Real Data

Here, the data set was sourced from Meintanis [17]. The data set in table 1 shows data from football (soccer) games where at least one kick goal was scored by the home team and at least one goal was scored directly from a penalty kick, foul kick, or many other direct kicks (all of them will be referred to as kick goals by any team). Here Z_1 stands for the time in minutes of the first kick goal scored by any team. Z_2 is the first goal of any kind that the home team has ever scored. Clearly all possibilities are open $Z_1 > Z_2$, $Z_2 > Z_1$ and $Z_2 = Z_1$.

Table 1. The UEFA Champions League data for the year 2004:2005 and 2005:2006

Z_1	Z_2	Z_1	Z_2	Z_1	Z_2	Z_1	Z_2
26	20	82	48	34	34	25	14
63	18	72	72	53	39	55	11
19	19	66	62	54	7	49	44
66	85	25	9	51	28	24	24
40	40	41	3	76	64	44	30
49	49	16	75	64	15	27	27
8	8	18	18	26	48	28	28
69	71	22	14	16	16	2	2
39	39	36	52	44	13		

From this data, we can determine the values of the unknown parameters and the confidence intervals listed in Table 2.

Table 2. The Confidence interval for the BOGE-G distribution

Parameter	Estimated interval	Confidence interval
β_1	0.64	[0.348, 0.931]
β_2	0.945	[0.348,1.458]
β_3	0.778	[0.422,1.133]
α	1.234	[-0.015,0.115]
λ	1.464	[1.339,1.529]
c	1	[0.935,1.065]

In order to demonstrate that the BOGE-G distribution is suitable for use as a lifespan model, we compare it to the bivariate exponentiated generalized weibull gompertz (BEGWG) distribution and the bivariate exponentiated modified weibull (BEMW) distribution. This data set is fitted using the BOGE-G model. Table 3. lists the values for the test statistics for the log likelihood (L), Akaike information criterion (AIC), correct Akaike information criterion (CAIC), Bayesian information criterion (BIC), and Hannan-Quinn information criterion (HQIC).

From Table 3. Because the BOGE-G distribution has the lowest AIC, CAIC, BIC, and HQIC test values, we may declare that it is

Table 3. L,AIC, CAIC, BIC and HQIC

The Model	L	AIC	CAIC	BIC	HQIC
BEGWG	-442.821	897.642	900.642	906.974	900.863
BEMW	-256.423	524.846	527.846	534.179	528.068
BOGE-G	-61.701	135.402	138.402	144.734	138.62

the best distribution.

6.2 The Second Real Data

The data set displays league information for American football (National Football League). It is formed from the games that were played over three weekends in 1986. The data was initially published in "Washington Post," and it is also accessible in [4]. Here, Z_1"the game time" refers to the first points scored by kicking the ball between the goal posts, while Z_2 "the game time" refers to the first points scored by moving the ball into the end zone, denoted by Z_1 and Z_2, respectively. These times are useful for casual viewers who want to know how long they will have to wait to see a touchdown or for viewers who are just interested at the beginning stage of a game.

The data (scoring time in minutes and seconds) are shown in table 4. In addition, all of the data points are split by 100 just for computation. Clearly all possibilities are open $Z_1 > Z_2$ means that the first score is an unconverted touchdown are safety, $Z_2 > Z_1$means that the first score is a field goal and $Z_2 = Z_1$means the first score is a converted touchdown.

Table 4. American Football (National Football League) league data

Z_1	Z_2	Z_1	Z_2	Z_1	Z_2	Z_1	Z_2
2.05	3.98	8.53	14.57	2.90	2.90	1.38	1.38
9.05	9.05	31.13	49.88	7.02	7.02	10.53	10.53
0.85	0.85	14.58	20.57	6.42	6.42	12.13	12.13
3.43	3.43	5.78	10.40	8.98	8.98	14.58	14.58
7.78	7.78	13.80	49.75	10.15	10.15	11.82	11.82
10.57	14.28	7.25	7.25	8.87	8.87	5.52	11.27
7.05	7.05	4.25	4.25	10.40	10.25	19.65	10.70
2.58	2.58	1.65	1.65	2.98	2.98	17.83	17.83
7.23	9.68	6.42	15.08	3.88	6.43	10.85	38.07
6.85	34.58	4.22	9.48	0.75	0.75		
32.45	32.45	15.53	15.53	11.63	17.37		

From the above data, we can determine the values of the unknown parameters and the confidence intervals listed in Table 5.

Table 5. The Confidence interval for the BOGE-G distribution

Parameter	Estimated interval	Confidence interval
β_1	0.059	[-0.022, 0.14]
β_2	0.372	[-0.022,0.556]
β_3	0.744	[0.5,0.988]
α	0.05	[-0.043,0.143]
λ	2	[1.907,2.093]
c	1	[0.907,1.093]

In order to demonstrate that the BOGE-G distribution is suitable for use as a lifespan model, we compare it to the bivariate exponentiated generalized weibull gompertz (BEGWG) distribution and the bivariate exponentiated modified weibull (BEMW) distribution. This data set is fitted using the BOGE-G model. Table 6. lists the values for the test statistics for the log likelihood (L), Akaike information criterion (AIC), correct Akaike information criterion (CAIC), Bayesian information criterion (BIC), and Hannan-Quinn information criterion (HQIC).

Table 6. L,AIC, CAIC, BIC and HQIC

The Model	L	AIC	CAIC	BIC	HQIC
BEGWG	-485.972	983.944	986.344	994.37	987.766
BEMW	-242.273	496.546	498.946	506.972	500.368
BOGE-G	-236.704	485.409	487.809	495.835	489.23

From table 6. The BOGE-G distribution has the lowest values for the AIC, CAIC, BIC, and HQIC tests, hence we can declare that it is the best distribution.

7 Simulation Study

In this section, we use simulation to eassess the performance of the MLEs for the BOGE-G parameters. For different combinations of φ sample sizes n=(20,50,100 and 200) are generated from the BOGE-G model. The population parameters are generated using the software Mathcad. In Table 7, the empirical findings are presented. It is clear that the estimates for these sample sizes are fairly consistent and near to the actual values of the parameters. Furthermore, the biases and standard errors of the MLEs diminish as sample size increases as expected. This study presents an assessment of the properties for MLE in terms of bias and mean square error (MSE). The following algorithm shows how to generate data from the BOGE-G distribution.

- Generate v_1, v_2 and v_3 from $v(0, 1)$.
- Compute $U_i = \left(\frac{1}{c}\right) \ln \left\{1 + \frac{c}{\lambda} \ln \left[1 - \frac{1}{\alpha} \ln (1 - v)^{\frac{1}{\beta}}\right]\right\}$,

  where $i = 1, 2, 3$.
- Obtain $Z_1 = \max\{U_1, U_3\}$ and $Z_2 = \max\{U_2, U_3\}$.

The MLEs values are listed in Table 7 for the BOGE-G distribution when $(\alpha, \lambda, c, \beta_1,\beta_2,\beta_3) = (0.7, 0.7, 0.7, 3, 3.4, 3.5)$ based on complete data.

Table 7. Estimation summaries for the BOGE-G distribution, for selected values of $\lambda = \alpha = c = 0.7$, $\beta_1 = 3$, $\beta_2 = 3.4$ and $\beta_3 = 3.5$

n=20		n=50		n=100		n=200	
Bias	**MSE**	**Bias**	**MSE**	**Bias**	**MSE**	**Bias**	**MSE**
-1.217	2.747	-0.641	0.417	-0.621	0.606	-0.533	0.354
-3.112	12.659	-1.5	2.603	-1.833	5.322	-1.174	1.544
-1.584	4.285	-1.71	3.318	-1.199	1.619	-1.117	1.435
0.018	0.037	0.07	0.014	0.118	0.049	0.131	0.045
0.332	0.11	0.294	0.086	0.31	0.096	0.291	0.085
-0.129	0.023	-0.054	0.0056	-0.1	0.016	-0.003	0.0023

From Table 7. , the following observations can be noted:

- The MSEs for the MLE always decrease to zero when n grows.
- The magnitude of bias in general always close to zero when n grows.

8 Conclusions

In this paper we have introduced a new model, called the bivariate Odd Generalized Exponential Gompertz (BOGE-G) distribution whose marginals are odd generalized exponential gompertz distributions.We also examined some of the new bivariate distribution's statistical features. Based on complete data, the parameters were estimated using the maximum likelihood technique, and it was founded that the MLEs performed quite well in estimating the parameters. We provided the observed Fisher information matrix. We derived the confidence interval estimates of the parameters using the maximum likelihood method . Two real data sets were used to demonstrate the usefulness of the proposed model, and it was founded that the new model provide a better fit than other sub models like the bivariate exponentiated generalized weibull gompertz (BEGWG) distribution and the bivariate exponentiated modified weibull (BEMW) distribution. We anticipate that the suggested model will find widespread use in disciplines like biology, gerontology, computer science, marketing, network theory, and others.

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Appendix

The approximate confidence intervals of the parameters based on the asymptotic distributions of their MLEs are derived. For the observed information matrix $\alpha, \lambda, c, \beta_{1,}\beta_2$ and β_3, we find second partial derivatives with the following notations as follows:

- $A_i = e^{\frac{\lambda}{c}(e^{cz_{1i}}-1)} - e^{-\alpha[e^{\frac{\lambda}{c}(e^{cz_{1i}-1})}-1]}$,
- $B_i = \frac{-\lambda}{c^2}e^{(cz_{1i}-1)} + \frac{\lambda}{c}(z_{1i}e^{cz_{1i}})$,
- $C_i = 1-\alpha\left(e^{\frac{\lambda}{c}(e^{cz_{1i}}-1)}-1\right) - e^{-\alpha[e^{\frac{\lambda}{c}(e^{cz_{1i}-1})}-1]}$,
- $D_i = 1-\alpha e^{\frac{\lambda}{c}(e^{cz_{1i}}-1)} - e^{-\alpha[e^{\frac{\lambda}{c}(e^{cz_{1i}-1})}-1]}$,
- $E_i = \frac{2\lambda}{c^3}(e^{cz_{1i}}-1) - \frac{2\lambda}{c^2}z_{1i}e^{cz_{1i}} + \frac{\lambda}{c}z_{1i}e^{cz_{1i}}$.

$$\begin{aligned}
\frac{\partial^2 l}{\partial\beta_1^2} &= I_{11} = \frac{-n_1}{(\beta_1+\beta_3)^2} - \frac{n_2}{(\beta_1)^2},\\
\frac{\partial^2 l}{\partial\beta_1\partial\beta_2} &= I_{12} = 0,\\
\frac{\partial^2 l}{\partial\beta_1\partial\beta_3} &= I_{13} = \frac{-n_1}{(\beta_1+\beta_3)^2}
\end{aligned}$$

$$\begin{aligned}
\frac{\partial^2 l}{\partial\beta_1\partial\alpha} &= I_{14} = \sum_{i=1}^{n_1}\frac{(W_{1i}-1)\,H_{1i}}{1-H_{1i}}\\
&+\sum_{i=1}^{n_2}\frac{(W_{2i}-1)\,H_{2i}}{1-H_{2i}}\\
&+\sum_{i=1}^{n_3}\frac{(W_{1i}-1)\,H_{1i}}{1-H_{1i}}
\end{aligned}$$

$$\begin{aligned}
\frac{\partial^2 l}{\partial\beta_1\partial\lambda} &= I_{15} = \frac{\alpha}{c}\sum_{i=1}^{n_1}\frac{(e^{cz_{1i}}-1)W_{1i}H_{1i}}{1-H_{1i}}\\
&+\frac{\alpha}{c}\sum_{i=1}^{n_2}\frac{(e^{cz_{2i}}-1)W_{2i}H_{2i}}{1-H_{2i}}\\
&+\frac{\alpha}{c}\sum_{i=1}^{n_3}\frac{(e^{cz_{1i}}-1)W_{1i}H_{1i}}{1-H_{1i}}
\end{aligned}$$

$$\begin{aligned}
\frac{\partial^2 l}{\partial\beta_1\partial c} &= I_{16}\\
&= \alpha\sum_{i=1}^{n_1}\frac{W_{1i}H_{1i}\left[e^{\frac{-\lambda}{c^2}(e^{cz_{1i}-1})}+\frac{\lambda}{c}(z_{1i}e^{cz_{1i}})\right]}{1-H_{1i}}\\
&+\alpha\sum_{i=1}^{n_2}\frac{W_{2i}H_{2i}\left[e^{\frac{-\lambda}{c^2}(e^{cz_{2i}-1})}+\frac{\lambda}{c}(z_{2i}e^{cz_{2i}})\right]}{1-H_{2i}}\\
&+\alpha\sum_{i=1}^{n_3}\frac{W_{1i}H_{1i}\left[e^{\frac{-\lambda}{c^2}(e^{cz_{1i}-1})}+\frac{\lambda}{c}(z_{1i}e^{cz_{1i}})\right]}{1-H_{1i}}
\end{aligned}$$

$$\begin{aligned}
\frac{\partial^2 l}{\partial\beta_2^2} &= I_{22} = \frac{-n_1}{(\beta_2)^2} - \frac{n_2}{(\beta_2+\beta_3)^2},\\
\frac{\partial^2 l}{\partial\beta_2\partial\beta_3} &= I_{23} = \frac{-n_2}{(\beta_2+\beta_3)^2}
\end{aligned}$$

$$\begin{aligned}
\frac{\partial^2 l}{\partial\beta_2\partial\alpha} &= I_{24} = \sum_{i=1}^{n_2}\frac{[W_{2i}-1]H_{2i}}{1-H_{2i}}\\
&+\sum_{i=1}^{n_2}\frac{[W_{1i}-1]H_{1i}}{1-H_{1i}}\\
&+\sum_{i=1}^{n_3}\frac{[W_{1i}-1]H_{1i}}{1-H_{1i}}
\end{aligned}$$

$$\begin{aligned}
\frac{\partial^2 l}{\partial\beta_2\partial\lambda} &= I_{25} = \frac{\alpha}{c}\sum_{i=1}^{n_1}\frac{(e^{cz_{2i}}-1)W_{2i}H_{2i}}{1-H_{2i}}\\
&+\frac{\alpha}{c}\sum_{i=1}^{n_2}\frac{(e^{cz_{1i}}-1)W_{1i}H_{1i}}{1-H_{1i}}\\
&+\frac{\alpha}{c}\sum_{i=1}^{n_3}\frac{(e^{cz_{1i}}-1)W_{1i}H_{1i}}{1-H_{1i}}
\end{aligned}$$

$$\begin{aligned}
\frac{\partial^2 l}{\partial\beta_2\partial c} &= I_{26}\\
&= \alpha\sum_{i=1}^{n_1}\frac{W_{2i}H_{2i}\left[e^{\frac{-\lambda}{c^2}(e^{cz_{2i}-1})}+\frac{\lambda}{c}(z_{2i}e^{cz_{2i}})\right]}{1-H_{2i}}\\
&+\alpha\sum_{i=1}^{n_2}\frac{W_{1i}H_{1i}\left[e^{\frac{-\lambda}{c^2}(e^{cz_{1i}-1})}+\frac{\lambda}{c}(z_{1i}e^{cz_{1i}})\right]}{1-H_{1i}}\\
&+\alpha\sum_{i=1}^{n_3}\frac{W_{1i}H_{1i}\left[e^{\frac{-\lambda}{c^2}(e^{cz_{1i}-1})}+\frac{\lambda}{c}(z_{1i}e^{cz_{1i}})\right]}{1-H_{1i}}
\end{aligned}$$

$$\frac{\partial^2 l}{\partial\beta_3^2} = I_{33} = \frac{-n_1}{(\beta_1+\beta_3)^2} - \frac{n_2}{(\beta_1+\beta_3)^2} - \frac{n_3}{(\beta_3)^2}$$

$$\frac{\partial^2 l}{\partial\beta_3\partial\alpha} = I_{34} = \sum_{i=1}^{n_1}\frac{(W_{1i}-1)H_{1i}}{1-H_{1i}} + \sum_{i=1}^{n_2}\frac{(W_{2i}-1)H_{2i}}{1-H_{2i}} + \sum_{i=1}^{n_3}\frac{(W_{1i}-1)H_{1i}}{1-H_{1i}}$$

$$\frac{\partial^2 l}{\partial\beta_3\partial\lambda} = I_{35} = \frac{\alpha}{c}\sum_{i=1}^{n_1}\frac{(e^{cz_{1i}}-1)W_{1i}H_{1i}}{1-H_{1i}} + \frac{\alpha}{c}\sum_{i=1}^{n_2}\frac{(e^{cz_{2i}}-1)W_{2i}H_{2i}}{1-H_{2i}} + \frac{\alpha}{c}\sum_{i=1}^{n_3}\frac{(e^{cz_{1i}}-1)W_{1i}H_{1i}}{1-H_{1i}}$$

$$\begin{aligned}\frac{\partial^2 l}{\partial\beta_3\partial c} &= I_{36}\\ &= \alpha\sum_{i=1}^{n_1}\frac{W_{1i}H_{1i}\left[e^{\frac{-\lambda}{c^2}(e^{cz_{1i}}-1)}+\frac{\lambda}{c}(z_{1i}e^{cz_{1i}})\right]}{1-H_{1i}}\\ &+\alpha\sum_{i=1}^{n_2}\frac{W_{2i}H_{2i}\left[e^{\frac{-\lambda}{c^2}(e^{cz_{2i}}-1)}+\frac{\lambda}{c}(z_{2i}e^{cz_{2i}})\right]}{1-H_{2i}}\\ &+\alpha\sum_{i=1}^{n_3}\frac{W_{1i}H_{1i}\left[e^{\frac{-\lambda}{c^2}(e^{cz_{1i}}-1)}+\frac{\lambda}{c}(z_{1i}e^{cz_{1i}})\right]}{1-H_{1i}}\end{aligned}$$

$$\begin{aligned}\frac{\partial^2 l}{\partial\alpha^2} &= I_{44}\\ &= \frac{-2n_1}{\alpha^2}-(\beta_1+\beta_3-1)\sum_{i=1}^{n_1}\frac{(W_{1i})^2H_{1i}}{(1-H_{1i})^2}\\ &-(\beta_2-1)\sum_{i=1}^{n_1}\frac{(W_{2i})^2H_{2i}}{(1-H_{2i})^2}-\frac{2n_2}{\alpha^2}-(\beta_2+\beta_3-1)\\ &\sum_{i=1}^{n_2}\frac{(W_{1i})^2H_{1i}}{(1-H_{1i})^2}-(\beta_1-1)\sum_{i=1}^{n_2}\frac{(W_{2i})^2H_{2i}}{(1-H_{2i})^2}\\ &-\frac{n_3}{\alpha^2}-(\beta_1+\beta_2+\beta_3-1)\sum_{i=1}^{n_3}\frac{(W_{1i})^2H_{1i}}{(1-H_{1i})^2}\end{aligned}$$

$$\begin{aligned}\frac{\partial^2 l}{\partial\alpha\partial\lambda} &= I_{45} = \frac{-1}{c}\sum_{i=1}^{n_1}(e^{cz_{1i}}-1)W_{1i}\\ &+\frac{(\beta_1+\beta_3-1)}{c}\sum_{i=1}^{n_1}\frac{(e^{cz_{1i}}-1)A_iC_i}{(1-H_{1i})^2}\\ &+\frac{(\beta_2-1)}{c}\sum_{i=1}^{n_1}\frac{(e^{cz_{2i}}-1)A_iC_i}{(1-H_{2i})^2}\\ &+\frac{(\beta_2+\beta_3-1)}{c}\sum_{i=1}^{n_2}\frac{(e^{cz_{1i}}-1)A_iC_i}{(1-H_{1i})^2}\\ &+\frac{(\beta_1-1)}{c}\sum_{i=1}^{n_2}\frac{(e^{cz_{2i}}-1)W_{2i}A_iC_i}{(1-H_{2i})^2}\\ &+\frac{(\beta_1+\beta_2+\beta_3-1)}{c}\sum_{i=1}^{n_3}\frac{(e^{cz_{1i}}-1)A_iC_i}{(1-H_{1i})^2}\end{aligned}$$

$$\begin{aligned}\frac{\partial^2 l}{\partial\lambda^2} &= I_{55} = \frac{-2n_1}{\lambda^2}-\frac{\alpha}{c^2}\sum_{i=1}^{n_1}(e^{cz_{1i}}-1)^2W_{1i}\\ &+\frac{\alpha(\beta_1+\beta_3-1)}{c^2}\sum_{i=1}^{n_1}\frac{(e^{cz_{1i}}-1)^2A_iD_i}{(1-H_{1i})^2}\\ &-\frac{\alpha}{c^2}\sum_{i=1}^{n_1}(e^{cz_{2i}}-1)^2W_{2i}+\frac{\alpha(\beta_2-1)}{c^2}\\ &\sum_{i=1}^{n_1}\frac{(e^{cz_{2i}}-1)^2A_iD_i}{(1-H_{2i})^2}-\frac{2n_2}{\lambda^2}\\ &-\frac{\alpha}{c^2}\sum_{i=1}^{n_2}(e^{cz_{1i}}-1)^2W_{1i}+\frac{\alpha(\beta_2+\beta_3-1)}{c^2}\\ &\sum_{i=1}^{n_2}\frac{(e^{cz_{1i}}-1)^2A_iD_i}{(1-H_{1i})^2}-\frac{\alpha}{c^2}\sum_{i=1}^{n_2}(e^{cz_{2i}}-1)^2\\ &W_{2i}+\frac{\alpha(\beta_1-1)}{c^2}\sum_{i=1}^{n_2}\frac{(e^{cz_{2i}}-1)^2A_iD_i}{(1-H_{2i})^2}\\ &-\frac{n_1}{\lambda^2}-\frac{\alpha}{c^2}\sum_{i=1}^{n_3}(e^{cz_{1i}}-1)^2W_{1i}\\ &+\frac{\alpha(\beta_1+\beta_2+\beta_3-1)}{c^2}\\ &\sum_{i=1}^{n_3}\frac{(e^{cz_{1i}}-1)^2A_iD_i}{(1-H_{1i})^2}\end{aligned}$$

$$\begin{aligned}\frac{\partial^2 l}{\partial\alpha\partial c} &= I_{46} = -\sum_{i=1}^{n_1}B_iW_{1i}+(\beta_1+\beta_3-1)\sum_{i=1}^{n_1}\frac{A_iB_iC_i}{(1-H_{1i})^2}\\ &-\sum_{i=1}^{n_1}B_iW_{2i}+(\beta_2-1)\sum_{i=1}^{n_1}\frac{A_iB_iC_i}{(1-H_{2i})^2}\\ &-\sum_{i=1}^{n_2}B_iW_{1i}+(\beta_2+\beta_3-1)\sum_{i=1}^{n_2}\frac{A_iB_iC_i}{(1-H_{1i})^2}\\ &-\sum_{i=1}^{n_2}B_iW_{2i}+(\beta_1-1)\sum_{i=1}^{n_2}\frac{A_iB_iC_i}{(1-H_{2i})^2}\end{aligned}$$

$$\frac{\partial^2 l}{\partial\lambda\partial c} = I_{56} = \frac{1}{\lambda}\sum_{i=1}^{n_1} B_i - \frac{\alpha}{c}\sum_{i=1}^{n_1} \left[\frac{-1}{c}(e^{cz_{1i}}-1) + z_{1i}e^{cz_{1i}} + B_i(e^{cz_{1i}}-1)\right] W_{1i} + \frac{1}{\lambda}\sum_{i=1}^{n_1} B_i + \frac{\alpha(\beta_1+\beta_3-1)}{\lambda} \sum_{i=1}^{n_1} \frac{\left[1-H_{1i}+\frac{\lambda}{c}(e^{cz_{1i}}-1)D_i\right] A_i B_i}{(1-H_{1i})^2} - \frac{\alpha}{c}\sum_{i=1}^{n_1}\left[\begin{array}{c} \frac{-1}{c}(e^{cz_{2i}}-1) + z_{2i}e^{cz_{2i}} \\ +B_i(e^{cz_{2i}}-1) \end{array}\right] W_{2i} + \frac{\alpha(\beta_2-1)}{\lambda}\sum_{i=1}^{n_1} \frac{\begin{array}{c}\left[1-H_{2i}+\frac{\lambda}{c}(e^{cz_{2i}}-1)D_i\right] \\ \times A_i B_i\end{array}}{(1-H_{2i})^2} + \frac{1}{\lambda}\sum_{i=1}^{n_2} B_i - \frac{\alpha}{c}\sum_{i=1}^{n_2}\left[\begin{array}{c} \frac{-1}{c}(e^{cz_{1i}}-1) + z_{1i}e^{cz_{1i}} \\ +B_i(e^{cz_{1i}}-1) \end{array}\right] W_{1i} + \frac{\alpha(\beta_2+\beta_3-1)}{\lambda} \sum_{i=1}^{n_2} \frac{\left[1-H_{1i}+\frac{\lambda}{c}(e^{cz_{1i}}-1)D_i\right] A_i B_i}{(1-H_{1i})^2} + \frac{1}{\lambda}\sum_{i=1}^{n_2} B_i - \frac{\alpha}{c}\sum_{i=1}^{n_2}\left[\begin{array}{c} \frac{-1}{c}(e^{cz_{2i}}-1) + z_{2i}e^{cz_{2i}} \\ +B_i(e^{cz_{2i}}-1) \end{array}\right] W_{2i} + \frac{\alpha(\beta_1-1)}{\lambda}\sum_{i=1}^{n_2} \frac{\begin{array}{c}\left[1-H_{2i}+\frac{\lambda}{c}(e^{cz_{2i}}-1)D_i\right] \\ \times A_i B_i\end{array}}{(1-H_{2i})^2} + \frac{1}{\lambda}\sum_{i=1}^{n_3} B_i - \frac{\alpha}{c}\sum_{i=1}^{n_3}\left[\begin{array}{c} \frac{-1}{c}(e^{cz_{1i}}-1) + z_{1i}e^{cz_{1i}} \\ +B_i(e^{cz_{1i}}-1) \end{array}\right] W_{1i} + \frac{\alpha(\beta_1+\beta_2+\beta_3-1)}{\lambda} \sum_{i=1}^{n_3} \frac{\left[1-H_{1i}+\frac{\lambda}{c}(e^{cz_{1i}}-1)D_i\right] A_i B_i}{(1-H_{1i})^2}$$

$$\frac{\partial^2 l}{\partial c^2} = I_{66} = \sum_{i=1}^{n_1} E_i - \alpha\sum_{i=1}^{n_1}\left[E_i + B_i^2\right] W_{1i} + (\beta_1+\beta_3-1)\alpha \sum_{i=1}^{n_1} \frac{\left[B_i^2 D_i + E_i(1-H_{1i})\right] A_i}{(1-H_{1i})^2} + \sum_{i=1}^{n_1} E_i - \alpha\sum_{i=1}^{n_1}\left[E_i + B_i^2\right] W_{2i} + (\beta_2-1)\alpha \sum_{i=1}^{n_1} \frac{\left[B_i^2 D_i + E_i(1-H_{2i})\right] A_i}{(1-H_{2i})^2} + \sum_{i=1}^{n_2} E_i - \alpha\sum_{i=1}^{n_2}\left[E_i + B_i^2\right] W_{1i} + (\beta_2+\beta_3-1)\alpha \sum_{i=1}^{n_2} \frac{\left[B_i^2 D_i + E_i(1-H_{1i})\right] A_i}{(1-H_{1i})^2} + \sum_{i=1}^{n_2} E_i - \alpha\sum_{i=1}^{n_2}\left[E_i + B_i^2\right] W_{2i} + (\beta_1-1)\alpha \sum_{i=1}^{n_2} \frac{\left[B_i^2 D_i + E_i(1-H_{2i})\right] A_i}{(1-H_{2i})^2} + \sum_{i=1}^{n_3} E_i - \alpha\sum_{i=1}^{n_3}\left[E_i + B_i^2\right] W_{1i} + (\beta_1+\beta_2+\beta_3-1)\alpha \sum_{i=1}^{n_3} \frac{\left[B_i^2 D_i + E_i(1-H_{1i})\right] A_i}{(1-H_{1i})^2}$$