Lecture Outline:

1. IK Problem Formulation.
2. Geometrical Approach.
3. Algebraic Approach.
   - Inverse Position Problem.
   - Inverse Orientation Problem.
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1. IK Problem Formulation.
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IK Problem Formulation:

**Given**: Desired position and orientation of end-effector, \( p \).

**Required**: Joint Variables \( q \) (\( \theta \) or \( d \)) to get \( p \)

\[
q = f(p)
\]

Given the desired \( 4 \times 4 \) homogeneous transformation \( H \)

\[
H = \begin{bmatrix}
R^0_n & o^0_n \\
0 & 1
\end{bmatrix}
\]

The task is to find a solution (possibly one of many) of the equation:

\[
T^0_n(q_1, \ldots, q_n) = A_1(q_1) \ A_2(q_2) \ldots A_n(q_n) = H
\]

It is 12 equations with respect to \( n \) variables \( q_1, \ldots, q_n \)
IK Problem Formulation:

Given: Desired position and orientation of end-effector, \( \mathbf{p} \).

Required: Joint Variables \( \mathbf{q} \) (\( \theta \) or \( d \)) to get \( \mathbf{p} \)

\[ \mathbf{q} = f(\mathbf{p}) \]

It is often advantageous to find \( q_1, \ldots, q_n \) in a **closed** rather than **numerical** form:

\[ q_k = f_k(h_{11}, h_{12}, \ldots, h_{34}), \ k = 1, \ldots, n \]

This will allow fast and deterministic computation of the joints variables instead of searching for a possible solution.
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Geometrical Approach:

Example: Two link manipulator

<table>
<thead>
<tr>
<th>link</th>
<th>( a_i )</th>
<th>( \alpha_i )</th>
<th>( d_i )</th>
<th>( \theta_i )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>( a_1 )</td>
<td>0</td>
<td>0</td>
<td>( \theta_1 )</td>
</tr>
<tr>
<td>2</td>
<td>( a_2 )</td>
<td>0</td>
<td>0</td>
<td>( \theta_2 )</td>
</tr>
</tbody>
</table>
Geometrical Approach:

Example: Two link manipulator

\[
A_1 = \begin{bmatrix}
  c_1 & -s_1 & 0 & a_1c_1 \\
  s_1 & c_1 & 0 & a_1s_1 \\
  0 & 0 & 1 & 0 \\
  0 & 0 & 0 & 1
\end{bmatrix}, \quad
A_2 = \begin{bmatrix}
  c_2 & -s_2 & 0 & a_2c_2 \\
  s_2 & c_2 & 0 & a_2s_2 \\
  0 & 0 & 1 & 0 \\
  0 & 0 & 0 & 1
\end{bmatrix}
\]
Geometrical Approach:

Example: Two link manipulator

\[ T_2^0 = A_1A_2 = \begin{bmatrix} c_{12} & -s_{12} & 0 & a_1c_1 + a_2c_{12} \\ s_{12} & c_{12} & 0 & a_1s_1 + a_2s_{12} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \]

Forward Kinematics:

\[ x = a_1 \cos(\theta_1) + a_2 \cos(\theta_1 + \theta_2) \]
\[ y = a_1 \sin(\theta_1) + a_2 \sin(\theta_1 + \theta_2) \]
Geometrical Approach:

Example: Two link manipulator

\[
\begin{align*}
x &= a_1 \cos(\theta_1) + a_2 \cos(\theta_1 + \theta_2) \\
y &= a_1 \sin(\theta_1) + a_2 \sin(\theta_1 + \theta_2)
\end{align*}
\]

In Inverse Kinematics, we need the joint variables \( \theta_1, \theta_2 \) in terms of the given \( x \) and \( y \).

- The Forward Kinematic equation are \textbf{non-linear} of sine and cosine terms.
- It is not easy to find a solution or a unique solution in general.
Geometrical Approach:

Example: Two link manipulator

\[ c^2 = x^2 + y^2 \]

\[ c^2 = a_1^2 + a_2^2 - 2a_1a_2 \cos(180 - \theta_2) \]

\[ \cos(\theta_2) = \frac{x^2 + y^2 - a_1^2 - a_2^2}{2a_1a_2} := D \]
Geometrical Approach:

Example: Two link manipulator

If \( \cos \theta_2 \) is known,

\[
\sin(\theta_2) = \pm \sqrt{1 - D^2}
\]

So,

\[
\theta_2 = \tan^{-1} \left( \frac{\pm \sqrt{1 - D^2}}{D} \right)
\]

Two Solutions

where,

\[
D = \frac{x^2 + y^2 - a_1^2 - a_2^2}{2a_1a_2}
\]
Geometrical Approach:

Example: Two link manipulator

\[ \theta_1 = \phi - \alpha \]

\[ \theta_1 = \tan^{-1} \frac{y}{x} - \tan^{-1} \left( \frac{a_2 \sin \theta_2}{a_1 + a_2 \cos \theta_2} \right) \]
Geometrical Approach:

Example: Two link manipulator

\[
\theta_2 = \tan^{-1} \left( \pm \sqrt{1 - \frac{D^2}{D}} \right); \quad D = \frac{x^2 + y^2 - a_1^2 - a_2^2}{2a_1a_2}
\]

\[
\theta_1 = \tan^{-1} \frac{y}{x} - \tan^{-1} \frac{a_2 \sin \theta_2}{a_1 + a_2 \cos \theta_2}
\]

Two Solutions exist
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Algebraic Approach:

Example: RPP Robot

\[
T_3^0 = A_1 A_2 A_3 = \begin{bmatrix}
 c_1 & 0 & -s_1 & -s_1d_3 \\
 s_1 & 0 & c_1 & c_1d_3 \\
 0 & -1 & 0 & d_1 + d_2 \\
 0 & 0 & 0 & 1
\end{bmatrix}
\]

Forward Kinematics:

\[
x = -d_3 \sin \theta_1 \\
y = d_3 \cos \theta_1 \\
z = d_1 + d_2
\]
Algebraic Approach:

Example: RPP Robot

\[ x = -d_3 \sin \theta_1 \]
\[ y = d_3 \cos \theta_1 \]
\[ z = d_1 + d_2 \]

Inverse Kinematics:

\[ \theta_1 = \tan^{-1} \left( \frac{-x}{y} \right) \]
\[ d_2 = z - d_1 \]
\[ d_3 = \sqrt{x^2 + y^2} \]
Algebraic Approach:

Example: RP Robot

\[ A_2^0 = A_1^0 \times A_2^1 \]

\[
A_1^0 = \begin{bmatrix}
C\theta_1 & 0 & S\theta_1 & 0 \\
S\theta_1 & 0 & -C\theta_1 & 0 \\
0 & 1 & 0 & l_1 \\
0 & 0 & 0 & 1
\end{bmatrix}, \quad A_2^1 = \begin{bmatrix}
1 & 0 & S\theta_1 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & d_2 \\
0 & 0 & 0 & 1
\end{bmatrix}
\]

\[
A_2^0 = \begin{bmatrix}
C\theta_1 & 0 & S\theta_1 & d_2S\theta_1 \\
S\theta_1 & 0 & -C\theta_1 & -d_2C\theta_1 \\
0 & 1 & 0 & l_1 \\
0 & 0 & 0 & 1
\end{bmatrix}
\]

\[ x = d_2 \sin \theta_1 \]

\[ y = -d_2 \cos \theta_1 \]

\[ z = l_1 \]
Algebraic Approach:

Example: RP Robot

\[ x = d_2 \sin \theta_1 \]
\[ y = -d_2 \cos \theta_1 \]
\[ z = l_1 \]

Inverse Kinematics:

\[ \theta_1 = \tan^{-1} \left( \frac{-x}{y} \right) \]
\[ d_2 = \sqrt{x^2 + y^2} \]
Algebraic Approach:

Example: Stanford Manipulator

\[
T_6^0 = A_1 \cdots A_6
\]

\[
= \begin{bmatrix}
  r_{11} & r_{12} & r_{13} & d_x \\
  r_{21} & r_{22} & r_{23} & d_y \\
  r_{31} & r_{32} & r_{33} & d_z \\
  0 & 0 & 0 & 1
\end{bmatrix}
\]
Algebraic Approach:

**Example: Stanford Manipulator**

\[
T_6^0 = A_1 \cdots A_6 = \begin{bmatrix} r_{11} & r_{12} & r_{13} & d_x \\ r_{21} & r_{22} & r_{23} & d_y \\ r_{31} & r_{32} & r_{33} & d_z \\ 0 & 0 & 0 & 1 \end{bmatrix}
\]

Not easy to find the IK in direct form!

\[
\begin{align*}
r_{11} &= c_1 [c_2(c_4c_5c_6 - s_4s_6) - s_2s_5c_6] - d_2(s_4c_5c_6 + c_4s_6) \\
r_{21} &= s_1[c_2(c_4c_5c_6 - s_4s_6) - s_2s_5c_6] + c_1(s_4c_5c_6 + c_4s_6) \\
r_{31} &= -s_2(c_4c_5c_6 - s_4s_6) - c_2s_5c_6 \\
r_{12} &= c_1[-c_2(c_4c_5s_6 + s_4c_6) + s_2s_5s_6] - s_1(-s_4c_5s_6 + c_4c_6) \\
r_{22} &= -s_1[-c_2(c_4c_5s_6 + s_4c_6) + s_2s_5s_6] + c_1(-s_4c_5s_6 + c_4c_6) \\
r_{32} &= s_2(c_4c_5s_6 + s_4c_6) + c_2s_5s_6 \\
r_{13} &= c_1(c_2c_4s_5 + s_2c_5) - s_1s_4s_5 \\
r_{23} &= s_1(c_2c_4s_5 + s_2c_5) + c_1s_4s_5 \\
r_{33} &= -s_2c_4s_5 + c_2c_5 \\
d_x &= c_1s_2d_3 - s_1d_2 + d_6(c_1c_2c_4s_5 + c_1c_5s_2 - s_1s_4s_5) \\
d_y &= s_1s_2d_3 + c_1d_2 + d_6(c_1s_4s_5 + c_2c_4s_1s_5 + c_5s_1s_2) \\
d_z &= c_2d_3 + d_6(c_2c_5 - c_4s_2s_5)
\]
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Spherical Wrist:

- The joints in the kinematic chain between the arm and the end-effector are refereed as the wrist.
- The wrist joints are nearly always revolute.
Spherical Wrist:

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- It is common to attach a spherical wrist to the manipulator end to allow the orientation of the end-effector.
Spherical Wrist:

- The joints in the kinematic chain between the arm and the end-effector are referred to as the wrist.
- The wrist joints are nearly always revolute.
- It is common to attach a spherical wrist to the manipulator end to allow the orientation of the end-effector.
- In spherical wrist, the axes of the three joints are intersecting at the wrist center point.
Spherical Wrist:

- The joints in the kinematic chain between the arm and the end-effector are referred as the wrist.
- The wrist joints are nearly always revolute.
- It is common to attach a spherical wrist to the manipulator end to allow the orientation of the end-effector.
- In spherical wrist the axes of the three joints are intersecting at the wrist center point.
- The spherical wrist simplify the robot kinematics. It allow the decoupling of the position and orientation in the Inverse Kinematic analysis.
Kinematic Decoupling:

Given the desired end-effector pose: $R_6^0 \in SO(3)$ and $O_6^0 \in \mathbb{R}^3$

- Problem of inverse kinematics is quite difficult.
- If the manipulator has:
  - Six joints (DOF = 6).
  - The last 3 joint axes intersecting in one point (Spherical Wrist).

then the problem is decoupled into two sub-problems:

- Inverse position kinematics.
- Inverse orientation kinematics
**Kinematic Decoupling:**

Given the desired $4 \times 4$ homogeneous transformation $H$

$$H = \begin{bmatrix} R_6^0 & o_6 \\ 0 & 1 \end{bmatrix}$$

1. The spherical wrist assumption makes the position of the wrist center point $O_c$ is **independent on the end-effector orientation**.
2. The orientation of the end-effector **depends on the last three joints**.

**Inverse Position**: Solve the IK for the position of $O_c$.

**Inverse Rotation**: Solve the IK for the orientation $R_6^3$.

How to compute $O_c$ and $R_6^3$?
Kinematic Decoupling:

[1] Calculation of the wrist center $O_c$:

- The position of the end-effector center, $O_6$, is obtained by a translation of distance $d_6$ along $z_5$ from $O_c$. Since $z_5$ and $z_6$ are on the same axis, we can choose the third column of the desired rotation $R_{6}^{0}$ as the direction of $z_6$ and $z_5$ w.r.t. base frame.

$$O_6 = O_c + d_6 R \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \Rightarrow O_c = O_6 - d_6 R \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

Required

$$\begin{bmatrix} x_c \\ y_c \\ z_c \end{bmatrix} = \begin{bmatrix} O_x - d_6 r_{13} \\ O_y - d_6 r_{23} \\ O_z - d_6 r_{33} \end{bmatrix}$$

Given
Kinematic Decoupling:

[2] Calculation of the orientation matrix $R^3_6$:

- We know that:

$$R^0_6 = R^0_3 R^3_6$$

So,

$$R^3_6 = [R^0_3]^{-1} R^0_6 \implies [R^0_3]^T R^0_6$$

Since $R^0_3$ depends on the first three joints variable, we can find its value by solving the forward kinematics after knowing the three joints values.

Required $R^3_6 = [R^0_3]^T R^0_6$ Given
Kinematic Decoupling: (1) Inverse Position Problem

What is the value of $\theta_1$, $\theta_2$ and $\theta_3$ to get $O_c = \begin{bmatrix} x_c \\ y_c \\ z_c \end{bmatrix}$?
Kinematic Decoupling: (1) Inverse Position Problem

- Projection on the x-y plane:

\[ \theta_1 = \tan^{-1} \frac{y_c}{x_c} \quad \text{or} \quad \theta_1 = \pi + \tan^{-1} \frac{y_c}{x_c} \]
Kinematic Decoupling: (1) Inverse Position Problem

- Project the manipulator on the Link 1 and Link 2 plane:

\[
\cos(\theta_3) = \frac{s^2 + r^2 - a_2^2 - a_3^2}{2a_2a_3} := D
\]

\[
r^2 = x_c^2 + y_c^2; \quad s^2 = (z_c - d_1)^2
\]
Kinematic Decoupling: (1) Inverse Position Problem

- Project the manipulator on the Link 1 and Link 2 plane:

\[
\begin{align*}
\sin(\theta_3) &= \pm \sqrt{1 - D^2} \\
r^2 &= x_c^2 + y_c^2; \quad s^2 = (z_c - d_1)^2
\end{align*}
\]
Kinematic Decoupling: (1) Inverse Position Problem

Project the manipulator on the Link 1 and Link 2 plane:

\[ \theta_3 = \tan^{-1} \left( \pm \sqrt{1 - D^2} \right) \]

\[ D = \frac{s^2 + r^2 - a_2^2 - a_3^2}{2a_2a_3} \]

\[ s^2 = x_c^2 + y_c^2; \quad r^2 = (z_c - d_1)^2 \]
Kinematic Decoupling: (1) Inverse Position Problem

- Project the manipulator on the Link 1 and Link 2 plane:

\[ \theta_2 = \tan^{-1} \frac{s}{r} - \tan^{-1} \left( \frac{a_3 \sin \theta_3}{a_2 + a_3 \cos \theta_3} \right) \]

\[ r^2 = x_c^2 + y_c^2; \quad s^2 = (z_c - d_1)^2 \]
Kinematic Decoupling: (1) Inverse Position Problem

Inverse Position Problem:

\[
\theta_1 = \text{atan2}(y_c, x_c) \quad \text{OR} \quad \theta_1 = \text{atan2}(y_c, x_c) + \pi
\]

\[
\theta_3 = \text{atan2}(\pm \sqrt{1 - D^2}, D)
\]

\[
\theta_2 = \text{atan2}(s, r) - \text{atan2}(a_3 \sin \theta_3, a_2 + a_3 \cos \theta_3)
\]

where,

\[
r^2 = x_c^2 + y_c^2;
\]

\[
s^2 = (z_c - d_1)^2
\]

\[
D = \frac{s^2 + r^2 - a_2^2 - a_3^2}{2a_2a_3}
\]

What is the value of \( \theta_1, \theta_2 \) and \( \theta_3 \) to get

\[O_c = \begin{bmatrix} x_c \\ y_c \\ z_c \end{bmatrix} \]?
Kinematic Decoupling: (2) Inverse Orientation Problem

\[ R_3^0 = \text{Rotation}(H_3^0) \quad \text{From DH parameters} \]

<table>
<thead>
<tr>
<th>Link</th>
<th>( a_i )</th>
<th>( \alpha_i )</th>
<th>( d_i )</th>
<th>( \theta_i )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0</td>
<td>90</td>
<td>( d_1 )</td>
<td>( \theta_1^* )</td>
</tr>
<tr>
<td>2</td>
<td>( a_2 )</td>
<td>0</td>
<td>0</td>
<td>( \theta_2^* )</td>
</tr>
<tr>
<td>3</td>
<td>( a_3 )</td>
<td>0</td>
<td>0</td>
<td>( \theta_3^* )</td>
</tr>
</tbody>
</table>
Kinematic Decoupling: (2) Inverse Orientation Problem

\[ R_3^0 = \text{Rotation}(H_3^0) \quad \text{From DH parameters} \]

\[
R_3^0 = \begin{bmatrix}
    c_1 c_{23} & -c_1 s_{23} & s_1 \\
    s_1 c_{23} & -s_1 s_{23} & -c_1 \\
    s_{23} & c_{23} & 0
\end{bmatrix}
\]

\[ R_6^3 = [R_3^0]^{-1} R = [R_3^0]^T R \]

\( R \) is the given rotation matrix of the end-effector.
Kinematic Decoupling: (2) Inverse Orientation Problem

\[ R_6^3 = [R_3^0]^T R \]

\[
\begin{bmatrix}
  c_4 c_5 c_6 - s_4 s_6 \\
  s_4 c_5 c_6 + c_4 s_6 \\
  -s_5 c_6
\end{bmatrix}
\begin{bmatrix}
  c_4 s_5 \\
  -c_4 s_5 \\
  s_5 s_6
\end{bmatrix}
= 
\begin{bmatrix}
  c_1 c_{23} & s_1 c_{23} & s_{23} \\
  -c_1 s_{23} & -s_1 s_{23} & c_{23} \\
  s_{1} & -c_{1} & 0
\end{bmatrix}
\begin{bmatrix}
  r_{11} & r_{12} & r_{13} \\
  r_{21} & r_{22} & r_{23} \\
  r_{31} & r_{32} & r_{33}
\end{bmatrix}
\]

\[
c_4 s_5 = c_1 c_{23} r_{13} + s_1 c_{23} r_{23} + s_{23} r_{33}
\]

\[
s_4 s_5 = -c_1 s_{23} r_{13} - s_1 s_{23} r_{23} + c_{23} r_{33}
\]

\[
c_5 = s_1 r_{13} - c_1 r_{23}
\]
Kinematic Decoupling: (2) Inverse Orientation Problem

\[ c_4 s_5 = c_1 c_{23} r_{13} + s_1 c_{23} r_{23} + s_{23} r_{33} \]
\[ s_4 s_5 = -c_1 s_{23} r_{13} - s_1 s_{23} r_{23} + c_{23} r_{33} \]
\[ c_5 = s_1 r_{13} - c_1 r_{23} \]

Inverse Orientation Problem:

\[ \theta_5 = \text{atan2}(\pm \sqrt{1 - (s_1 r_{13} - c_1 r_{23})^2}, s_1 r_{13} - c_1 r_{23}) \]
\[ \theta_4 = \text{atan2}(-c_1 s_{23} r_{13} - s_1 s_{23} r_{23} + c_{23} r_{33}, c_1 c_{23} r_{13} + s_1 c_{23} r_{23} + s_{23} r_{33}) \]
\[ \theta_6 = \text{atan2}(s_1 r_{12} - c_1 r_{22}, -s_1 r_{11} + c_1 r_{21}) \]
Kinematic Decoupling: IK solution for the Elbow Manipulator

\[
\theta_1 = \text{atan2}(y_c, x_c) \quad \text{OR} \quad \theta_1 = \text{atan2}(y_c, x_c) + \pi
\]

\[
\theta_3 = \text{atan2}(\pm \sqrt{1 - D^2}, D)
\]

\[
\theta_2 = \text{atan2}(s, r) - \text{atan2}(a_3 \sin \theta_3, a_2 + a_3 \cos \theta_3)
\]

\[
\theta_5 = \text{atan2}(\pm \sqrt{1 - (s_1 r_{13} - c_1 r_{23})^2}, s_1 r_{13} - c_1 r_{23})
\]

\[
\theta_4 = \text{atan2}(-c_1 s_3 r_{13} - s_1 s_3 r_{23} + c_3 r_{33}, c_1 c_3 r_{13} + s_1 c_3 r_{23} + s_3 r_{33})
\]

\[
\theta_6 = \text{atan2}(s_1 r_{12} - c_1 r_{22}, -s_1 r_{11} + c_1 r_{21})
\]

where,

\[
r^2 = x_c^2 + y_c^2; \quad s^2 = (z_c - d_1)^2
\]

\[
D = \frac{s^2 + r^2 - a_2^2 - a_3^2}{2a_2 a_3}
\]

Four Solutions exist for the \(O_c\)!

Elbow Manipulator with Spherical Wrist
Kinematic Decoupling: IK solution for the Elbow Manipulator

Left Arm Elbow Up

Right Arm Elbow Up

Left Arm Elbow Down

Right Arm Elbow Down

Elbow Manipulator with Spherical Wrist
End of Lecture

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