ECE447: Robotics Engineering

Lecture 7: Inverse Kinematics

Dr. Haitham El-Hussieny

Electronics and Communications Engineering
Faculty of Engineering (Shoubra)
Benha University

Spring 2017

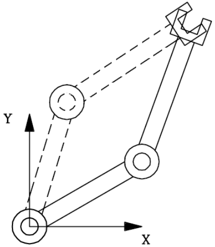
Lecture Outline:

1. IK Problem Formulation.
2. Geometrical Approach.
3. Algebraic Approach.
4. Kinematic Decoupling.
   - Inverse Position Problem.
   - Inverse Orientation Problem.

Table of Contents

IK Problem Formulation:

Inverse Kinematics



Given: Desired position and orientation of end-effector, **p**.
Required: Joint Variables **q** (θ or d) to get **p**

$$\mathbf{q} = f(\mathbf{p})$$

Given the desired 4×4 homogeneous transformation H

$$H = \begin{bmatrix} R_n^0 & o_n^0 \\ 0 & 1 \end{bmatrix}$$

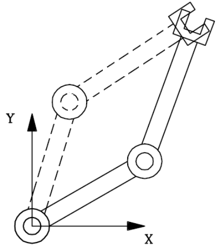
The task is to find a solution (possibly one of many) of the equation:

$$T_n^0(q_1, \ldots, q_n) = A_1(q_1)\ A_2(q_2) \ldots A_n(q_n) = H$$

It is 12 equations with respect to n variables $q_1, \ldots, q_n$

IK Problem Formulation:

Inverse Kinematics



Given: Desired position and orientation of end-effector, **p**.
Required: Joint Variables **q** (θ or d) to get **p**

$$\mathbf{q} = f(\mathbf{p})$$

It is often advantageous to find $q_1, \dots, q_n$ in a **closed rather than numerical** form:

$$q_k = f_k(h_{11}, h_{12}, \dots, h_{34}), \;\; k = 1, \dots, n$$

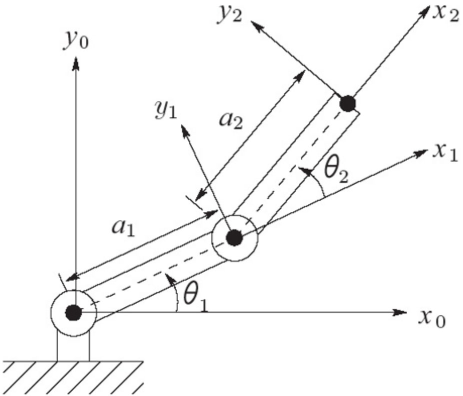
This will allow fast and deterministic computation of the joints variables instead of searching for a possible solution.

Table of Contents

Geometrical Approach:

Example: Two link manipulator

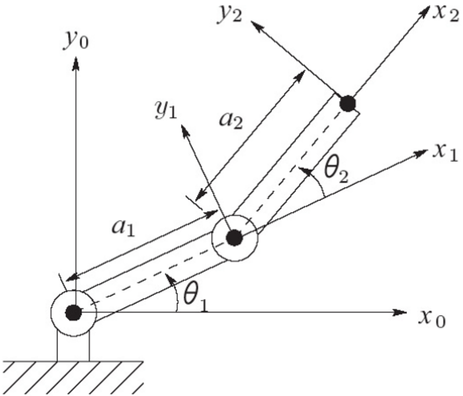
link	a_i	α_i	d_i	θ_i
1	a_1	0	0	θ_1
2	a_2	0	0	θ_2



Geometrical Approach:

Example: Two link manipulator

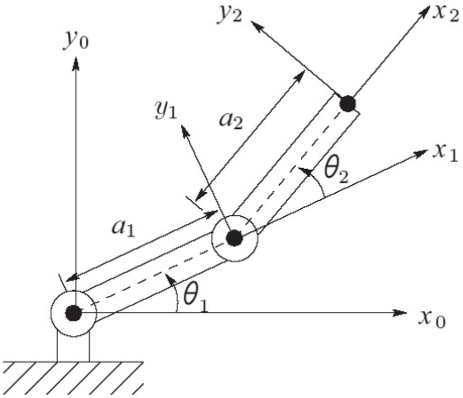
$$A_1 = \begin{bmatrix} c_1 & -s_1 & 0 & a_1c_1 \\ s_1 & c_1 & 0 & a_1s_1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, A_2 = \begin{bmatrix} c_2 & -s_2 & 0 & a_2c_2 \\ s_2 & c_2 & 0 & a_2s_2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$



Geometrical Approach:

Example: Two link manipulator

$$T_2^0 = A_1 A_2 = \begin{bmatrix} c_{12} & -s_{12} & 0 & a_1c_1 + a_2c_{12} \\ s_{12} & c_{12} & 0 & a_1s_1 + a_2s_{12} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$



Forward Kinematics:

$$x = a_1 cos(\theta_1) + a_2 cos(\theta_1 + \theta_2)$$

$$y = a_1 sin(\theta_1) + a_2 sin(\theta_1 + \theta_2)$$

Geometrical Approach:

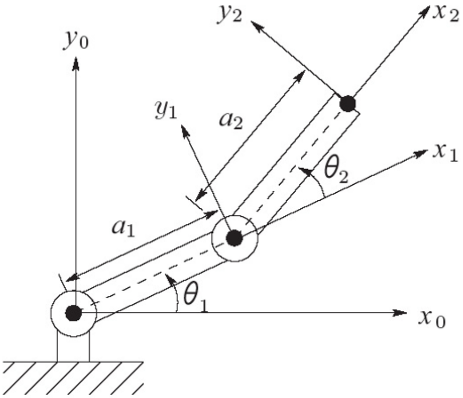
Example: Two link manipulator

$$x = a_1 cos(\theta_1) + a_2 cos(\theta_1 + \theta_2)$$

$$y = a_1 sin(\theta_1) + a_2 sin(\theta_1 + \theta_2)$$

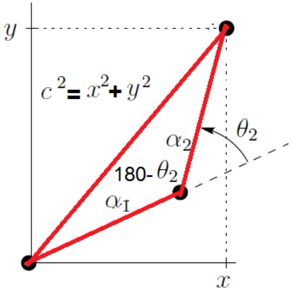
In Inverse Kinematics, we need the joint variables θ_1, θ_2 in terms of the given x and y.

- The Forward Kinematic equation are **non-linear** of sine and cosine terms.
- It is not easy to find a solution or a unique solution in general.



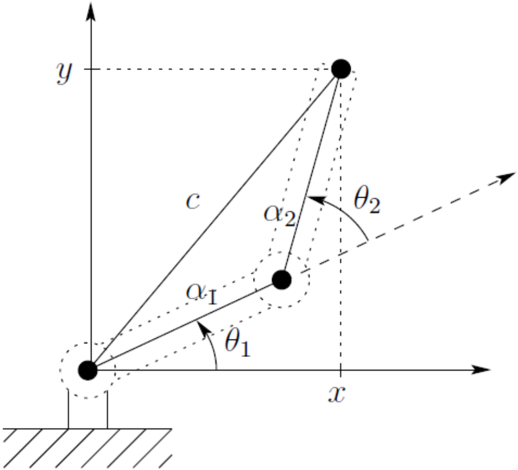
Geometrical Approach:

Example: Two link manipulator



$$c^2 = a_1^2 + a_2^2 - 2a_1a_2cos(180 - \theta_2)$$

$$cos(\theta_2) = \frac{x^2 + y^2 - a_1^2 - a_2^2}{2a_1a_2} := D$$



Geometrical Approach:

Example: Two link manipulator

If $cos\theta_2$ is known,

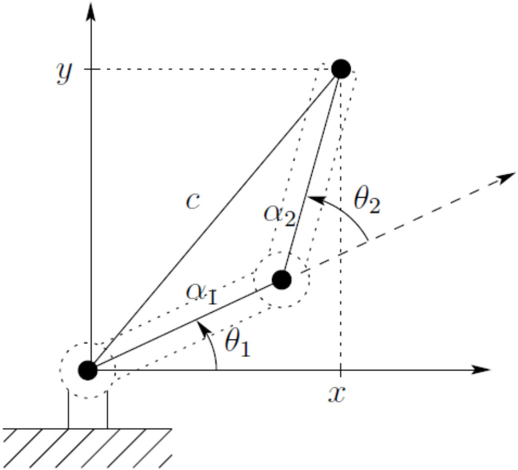
$$sin(\theta_2) = \pm\sqrt{1 - D^2}$$

So,

$$\boxed{\theta_2 = tan^{-1}\frac{\pm\sqrt{1 - D^2}}{D}}$$ Two Solutions

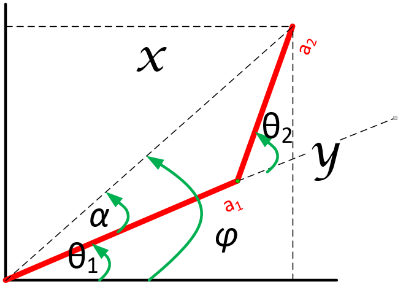
where,

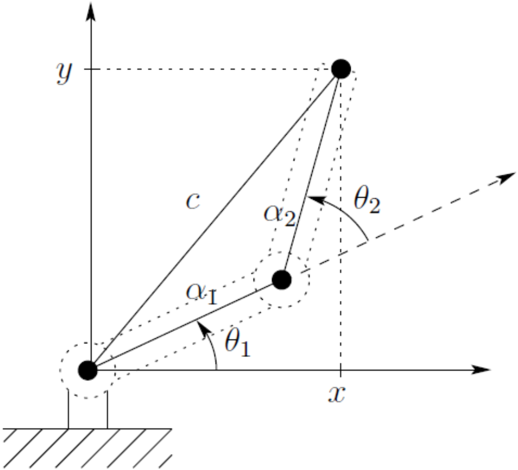
$$D = \frac{x^2 + y^2 - a_1^2 - a_2^2}{2a_1a_2}$$



Geometrical Approach:

Example: Two link manipulator





$$\theta_1 = \phi - \alpha$$

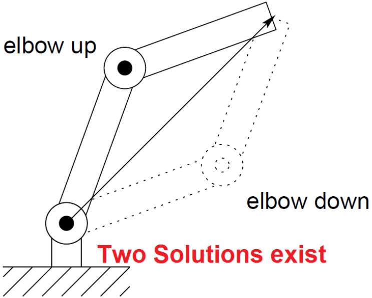
$$\theta_1 = tan^{-1}\frac{y}{x} - tan^{-1}\frac{a_2 sin\theta_2}{a_1 + a_2 cos\theta_2}$$

Geometrical Approach:

Example: Two link manipulator

$$\theta_2 = tan^{-1}\frac{\pm\sqrt{1-D^2}}{D}; \quad D = \frac{x^2+y^2-a_1^2-a_2^2}{2a_1a_2}$$

$$\theta_1 = tan^{-1}\frac{y}{x} - tan^{-1}\frac{a_2 sin\theta_2}{a_1 + a_2 cos\theta_2}$$



Two Solutions exist

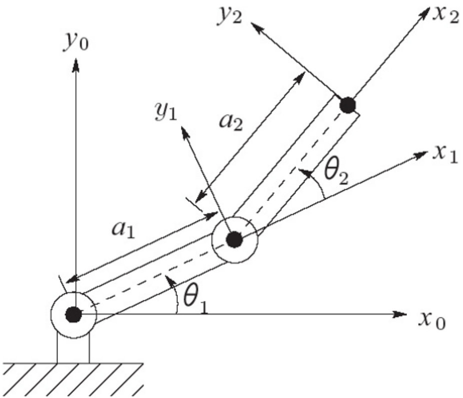
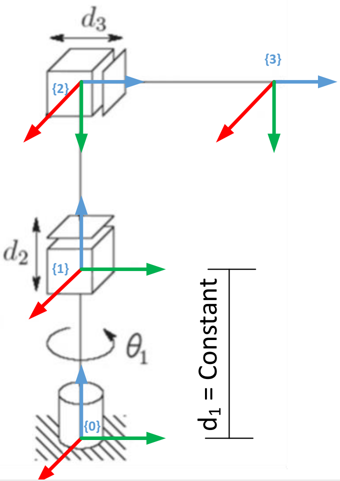


Table of Contents

Algebraic Approach:

Example: RPP Robot

$$T_3^0 = A_1 A_2 A_3 \quad = \begin{bmatrix} c_1 & 0 & -s_1 & -s_1 d_3 \\ s_1 & 0 & c_1 & c_1 d_3 \\ 0 & -1 & 0 & d_1 + d_2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$



Forward Kinematics:

$$x = -d_3 sin\theta_1$$

$$y = d_3 cos\theta_1$$

$$z = d_1 + d_2$$

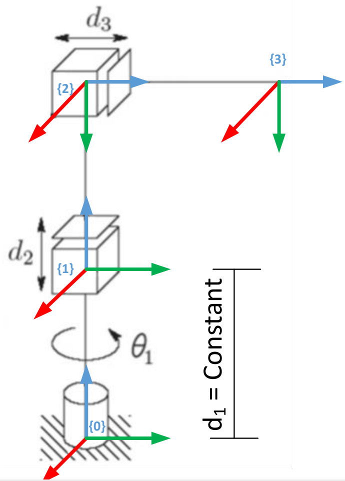
Algebraic Approach:

Example: RPP Robot

$$x = -d_3 sin\theta_1$$

$$y = d_3 cos\theta_1$$

$$z = d_1 + d_2$$



Inverse Kinematics:

$$\theta_1 = tan^{-1}\frac{-x}{y}$$

$$d_2 = z - d_1$$

$$d_3 = \sqrt{x^2 + y^2}$$

Algebraic Approach:

Example: RP Robot

$$A_2^0 = A_1^0 * A_2^1$$

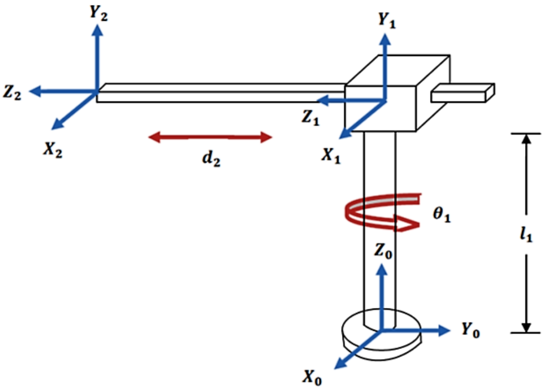
$$A_1^0 = \begin{bmatrix} C\theta_1 & 0 & S\theta_1 & 0 \\ S\theta_1 & 0 & -C\theta_1 & 0 \\ 0 & 1 & 0 & l_1 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad A_2^1 = \begin{bmatrix} 1 & 0 & S\theta_1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & d_2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$A_2^0 = \begin{bmatrix} C\theta_1 & 0 & S\theta_1 & d_2 S\theta_1 \\ S\theta_1 & 0 & -C\theta_1 & -d_2 C\theta_1 \\ 0 & 1 & 0 & l_1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$x = d_2 sin\theta_1$$

$$y = -d_2 cos\theta_1$$

$$z = l_1$$



Algebraic Approach:

Example: RP Robot

$$x = d_2 sin\theta_1$$

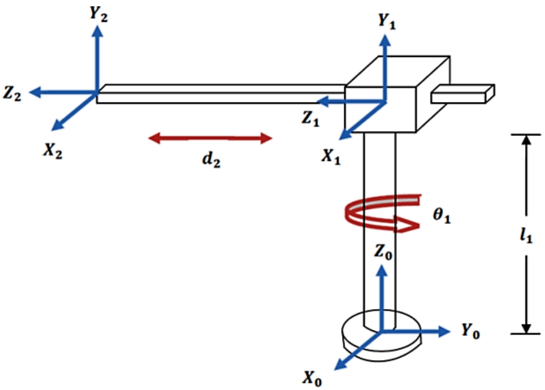
$$y = -d_2 cos\theta_1$$

$$z = l_1$$

Inverse Kinematics:

$$\theta_1 = tan^{-1}\frac{-x}{y}$$

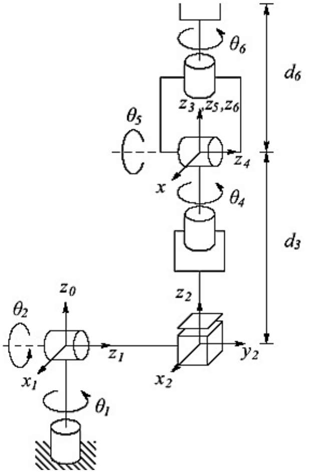
$$d_2 = \sqrt{x^2 + y^2}$$



Algebraic Approach:

Example: Stanford Manipulator

$$
\begin{aligned}
T_6^0 &= A_1 \cdots A_6 \\
&= \begin{bmatrix} r_{11} & r_{12} & r_{13} & d_x \\ r_{21} & r_{22} & r_{23} & d_y \\ r_{31} & r_{32} & r_{33} & d_z \\ 0 & 0 & 0 & 1 \end{bmatrix}
\end{aligned}
$$



Algebraic Approach:

Example: Stanford Manipulator

$$
\begin{aligned}
T_6^0 &= A_1 \cdots A_6 \\
&= \begin{bmatrix} r_{11} & r_{12} & r_{13} & d_x \\ r_{21} & r_{22} & r_{23} & d_y \\ r_{31} & r_{32} & r_{33} & d_z \\ 0 & 0 & 0 & 1 \end{bmatrix}
\end{aligned}
$$

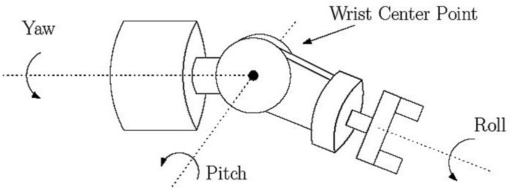
Not easy to find the IK in direct form!

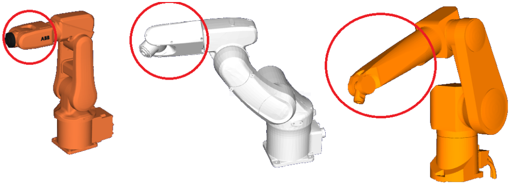
$$
\begin{aligned}
r_{11} &= c_1\left[c_2(c_4c_5c_6 - s_4s_6) - s_2s_5c_6\right] - d_2(s_4c_5c_6 + c_4s_6) \\
r_{21} &= s_1\left[c_2(c_4c_5c_6 - s_4s_6) - s_2s_5c_6\right] + c_1(s_4c_5c_6 + c_4s_6) \\
r_{31} &= -s_2(c_4c_5c_6 - s_4s_6) - c_2s_5c_6 \\
r_{12} &= c_1\left[-c_2(c_4c_5s_6 + s_4c_6) + s_2s_5s_6\right] - s_1(-s_4c_5s_6 + c_4c_6) \\
r_{22} &= -s_1\left[-c_2(c_4c_5s_6 + s_4c_6) + s_2s_5s_6\right] + c_1(-s_4c_5s_6 + c_4c_6) \\
r_{32} &= s_2(c_4c_5s_6 + s_4c_6) + c_2s_5s_6 \\
r_{13} &= c_1(c_2c_4s_5 + s_2c_5) - s_1s_4s_5 \\
r_{23} &= s_1(c_2c_4s_5 + s_2c_5) + c_1s_4s_5 \\
r_{33} &= -s_2c_4s_5 + c_2c_5 \\
d_x &= c_1s_2d_3 - s_1d_2 + +d_6(c_1c_2c_4s_5 + c_1c_5s_2 - s_1s_4s_5) \\
d_y &= s_1s_2d_3 + c_1d_2 + d_6(c_1s_4s_5 + c_2c_4s_1s_5 + c_5s_1s_2) \\
d_z &= c_2d_3 + d_6(c_2c_5 - c_4s_2s_5)
\end{aligned}
$$

Table of Contents

Spherical Wrist:

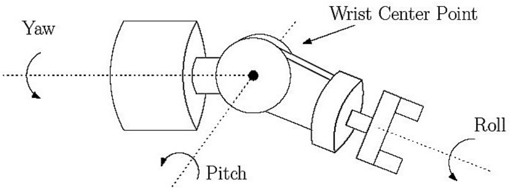
- The joints in the kinematic chain between the arm and the end-effector are refereed as the wrist.
- The wrist joints are nearly always revolute.

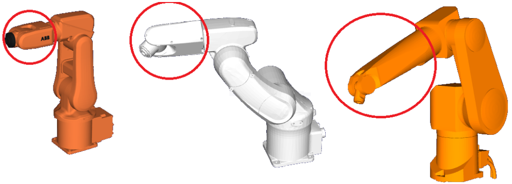




Spherical Wrist:

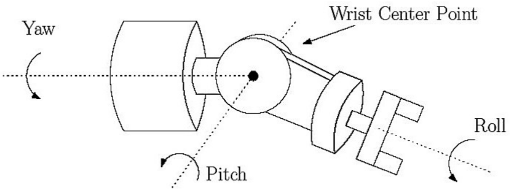
- The joints in the kinematic chain between the arm and the end-effector are refereed as the wrist.
- The wrist joints are nearly always revolute.
- It is common to attach a spherical wrist to the manipulator end to allow the orientation of the end-effector.

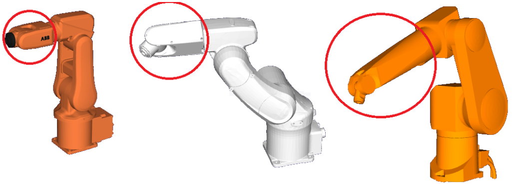




Spherical Wrist:

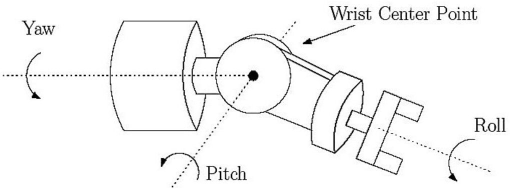
- The joints in the kinematic chain between the arm and the end-effector are refereed as the wrist.
- The wrist joints are nearly always revolute.
- It is common to attach a spherical wrist to the manipulator end to allow the orientation of the end-effector.
- In spherical wrist the axes of the three joints are intersecting at the wrist center point.

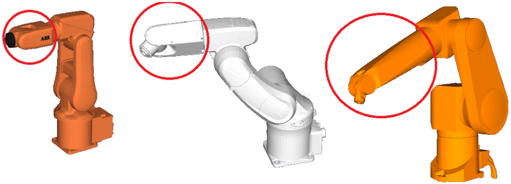




Spherical Wrist:

- The joints in the kinematic chain between the arm and the end-effector are refereed as the wrist.
- The wrist joints are nearly always revolute.
- It is common to attach a spherical wrist to the manipulator end to allow the orientation of the end-effector.
- In spherical wrist the axes of the three joints are intersecting at the wrist center point.
- The spherical wrist simplify the robot kinematics. It allow the decoupling of the position and orientation in the Inverse Kinematic analysis.





Kinematic Decoupling:

Given the desired end-effector pose: $R_6^0 \in SO(3)$ and $O_6^0 \in R^3$

- Problem of inverse kinematics is quite difficult.
- If the manipulator has:
  - Six joints (DOF = 6).
  - The last 3 joint axes intersecting in one point (**Spherical Wrist**).

  then the problem is decoupled into two sub-problems:
  - Inverse position kinematics.
  - Inverse orientation kinematics

Do fun with robots!

Kinematic Decoupling:

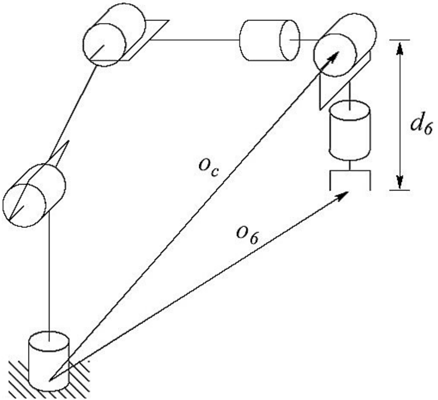
Given the desired 4×4 homogeneous transformation H

$$H = \begin{bmatrix} R_6^0 & o_6 \\ 0 & 1 \end{bmatrix}$$

1. The spherical wrist assumption makes the position of the wrist center point O_c is **independent on the end-effector orientation**.
2. The orientation of the end-effector **depends on the last three joints**.

Inverse Position: Solve the IK for the position of O_c.
Inverse Rotation: Solve the IK for the orientation R_6^3

How to compute O_c and R_6^3 ?

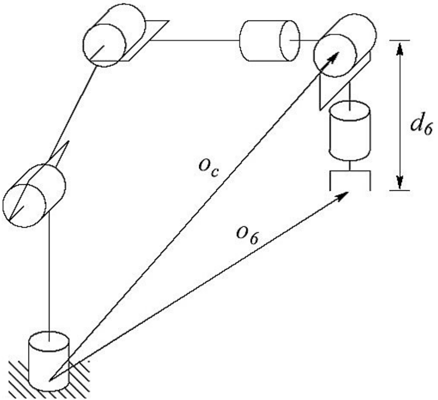


Elbow Manipulator with Spherical Wrist

Kinematic Decoupling:

[1] Calculation of the wrist center O_c:

- The position of the end-effector center, O_6, is obtained by a translation of distance d_6 along z_5 from O_c. Since z_5 and z_6 are on the same axis, we can choose the **third column of the desired rotation** R_6^0 as the direction of z_6 and z_5 w.r.t. base frame.



$$O_6 = O_c + d_6 R \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \Rightarrow O_c = O_6 - d_6 R \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

Required $\begin{bmatrix} x_c \\ y_c \\ z_c \end{bmatrix} = \begin{bmatrix} O_x - d_6 r_{13} \\ O_y - d_6 r_{23} \\ O_z - d_6 r_{33} \end{bmatrix}$ Given

Kinematic Decoupling:

[2] Calculation of the orientation matrix R_6^3:

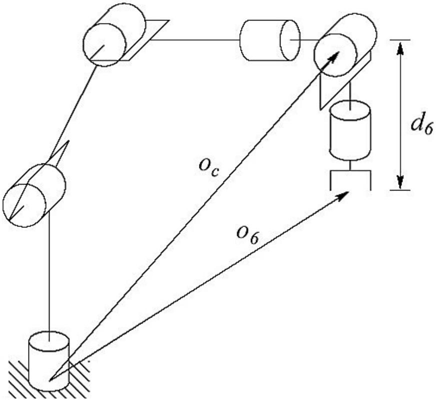
- We know that:

$$R_6^0 = R_3^0 R_6^3$$

So,

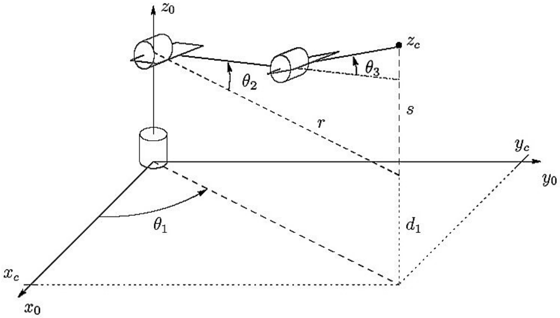
$$R_6^3 = [R_3^0]^{-1} \ R_6^0 \Rightarrow [R_3^0]^T \ R_6^0$$

Since R_3^0 **depends on the first three joints variable**, we can find its value by **solving the forward kinematics** after knowing the three joints values.



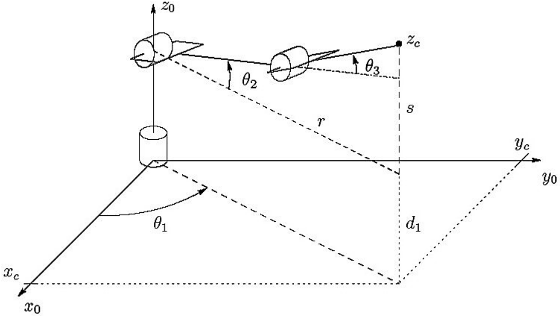
Required $R_6^3 = [R_3^0]^T \ R_6^0$ Given

Kinematic Decoupling: (1) Inverse Position Problem

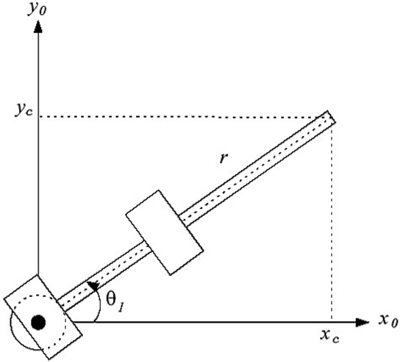


- What is the value of θ_1, θ_2 and θ_3 to get $O_c = \begin{bmatrix} x_c \\ y_c \\ z_c \end{bmatrix}$?

Kinematic Decoupling: (1) Inverse Position Problem

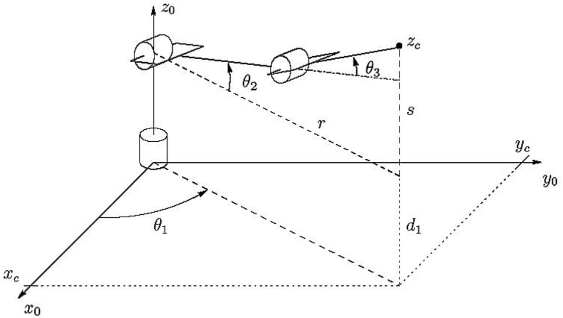


- Projection on the x-y plane:

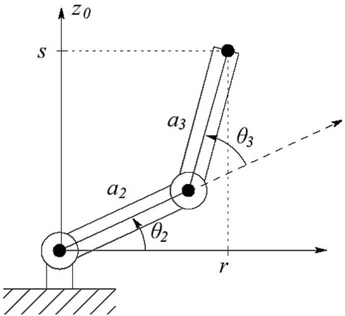


$$\theta_1 = tan^{-1}\frac{y_c}{x_c} \quad \text{or } \theta_1 = \pi + tan^{-1}\frac{y_c}{x_c}$$

Kinematic Decoupling: (1) Inverse Position Problem



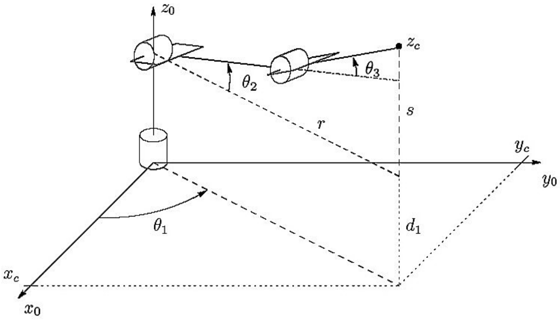
- Project the manipulator on the Link 1 and Link 2 plane:



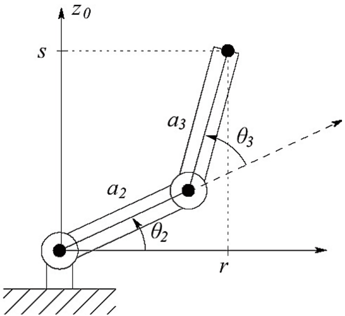
$$\cos(\theta_3) = \frac{s^2 + r^2 - a_2^2 - a_3^2}{2a_2a_3} := D$$

$$r^2 = x_c^2 + y_c^2; \qquad s^2 = (z_c - d_1)^2$$

Kinematic Decoupling: (1) Inverse Position Problem



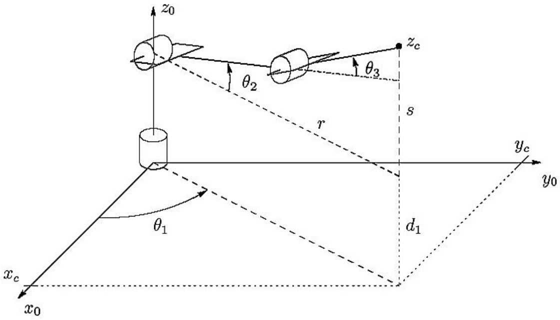
- Project the manipulator on the Link 1 and Link 2 plane:



$$\boxed{sin(\theta_3) = \pm\sqrt{1 - D^2}}$$

$$r^2 = x_c^2 + y_c^2; \qquad s^2 = (z_c - d_1)^2$$

Kinematic Decoupling: (1) Inverse Position Problem



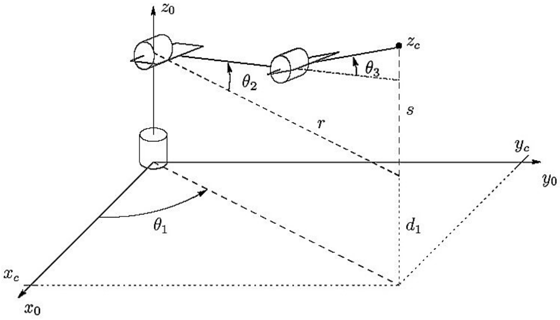
- Project the manipulator on the Link 1 and Link 2 plane:

$$\boxed{\theta_3 = tan^{-1}\frac{\pm\sqrt{1-D^2}}{D}}$$

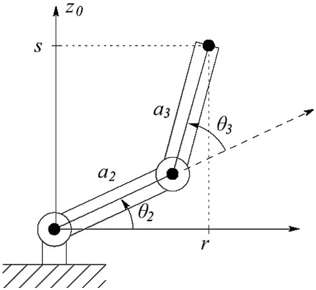
$$s^2 = x_c^2 + y_c^2; \qquad r^2 = (z_c - d_1)^2$$

$$D = \frac{s^2 + r^2 - a_2^2 - a_3^2}{2a_2a_3}$$

Kinematic Decoupling: (1) Inverse Position Problem



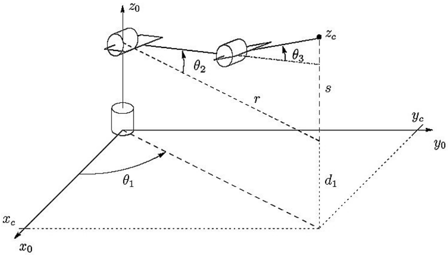
- Project the manipulator on the Link 1 and Link 2 plane:



$$\theta_2 = tan^{-1}\frac{s}{r} - tan^{-1}\frac{a_3 sin\theta_3}{a_2 + a_3 cos\theta_3}$$

$$r^2 = x_c^2 + y_c^2; \qquad s^2 = (z_c - d_1)^2$$

Kinematic Decoupling: (1) Inverse Position Problem



What is the value of θ_1, θ_2 and θ_3 to get

$$O_c = \begin{bmatrix} x_c \\ y_c \\ z_c \end{bmatrix} ?$$

Inverse Position Problem:

$$\boxed{\theta_1 = \text{atan2}(y_c, x_c) \quad \text{OR} \quad \theta_1 = \text{atan2}(y_c, x_c) + \pi}$$

$$\boxed{\theta_3 = \text{atan2}(\pm\sqrt{1 - D^2}, D)}$$

$$\boxed{\theta_2 = \text{atan2}(s, r) - \text{atan2}(a_3 sin\theta_3, a_2 + a_3 cos\theta_3)}$$

where,

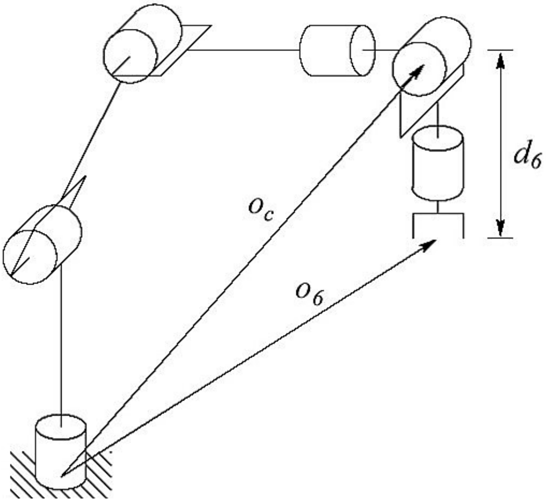
$$r^2 = x_c^2 + y_c^2; \qquad s^2 = (z_c - d_1)^2$$

$$D = \frac{s^2 + r^2 - a_2^2 - a_3^2}{2a_2a_3}$$

Kinematic Decoupling: (2) Inverse Orientation Problem

$R_3^0 = \text{Rotation}(H_3^0)$ From DH parameters

Link	a_i	α_i	d_i	θ_i
1	0	90	d_1	θ_1^*
2	a_2	0	0	θ_2^*
3	a_3	0	0	θ_3^*



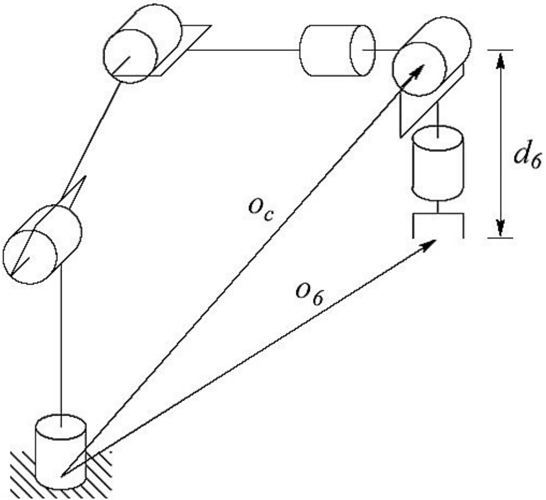
Kinematic Decoupling: (2) Inverse Orientation Problem

$R_3^0 = \text{Rotation}(H_3^0)$ From DH parameters

$$R_3^0 = \begin{bmatrix} c_1 c_{23} & -c_1 s_{23} & s_1 \\ s_1 c_{23} & -s_1 s_{23} & -c_1 \\ s_{23} & c_{23} & 0 \end{bmatrix}$$

$$R_6^3 = [R_3^0]^{-1} R = [R_3^0]^T R$$

R is the given rotation matrix of the end-effector.



Kinematic Decoupling: (2) Inverse Orientation Problem

$$R_6^3 = [R_3^0]^T R$$

$$\begin{bmatrix} c_4c_5c_6 - s_4s_6 & -c_4c_5s_6 - s_4c_6 & c_4s_5 \\ s_4c_5c_6 + c_4s_6 & -s_4c_5s_6 + s_4c_6 & c_4s_5 \\ -s_5c_6 & s_5s_6 & c_5 \end{bmatrix} = \begin{bmatrix} c_1c_{23} & s_1c_{23} & s_{23} \\ -c_1s_{23} & -s_1s_{23} & c_{23} \\ s_1 & -c_1 & 0 \end{bmatrix} \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix}$$

$$\begin{aligned} c_4s_5 &= c_1c_{23}r_{13} + s_1c_{23}r_{23} + s_{23}r_{33} \\ s_4s_5 &= -c_1s_{23}r_{13} - s_1s_{23}r_{23} + c_{23}r_{33} \\ c_5 &= s_1r_{13} - c_1r_{23} \end{aligned}$$

Kinematic Decoupling: (2) Inverse Orientation Problem

$$
\begin{aligned}
c_4 s_5 &= c_1 c_{23} r_{13} + s_1 c_{23} r_{23} + s_{23} r_{33} \\
s_4 s_5 &= -c_1 s_{23} r_{13} - s_1 s_{23} r_{23} + c_{23} r_{33} \\
c_5 &= s_1 r_{13} - c_1 r_{23}
\end{aligned}
$$

Inverse Orientation Problem:

$$\boxed{\theta_5 = \text{atan2}(\pm\sqrt{1 - (s_1 r_{13} - c_1 r_{23})^2}, s_1 r_{13} - c_1 r_{23})}$$

$$\boxed{\theta_4 = \text{atan2}(-c_1 s_{23} r_{13} - s_1 s_{23} r_{23} + c_{23} r_{33}, c_1 c_{23} r_{13} + s_1 c_{23} r_{23} + s_{23} r_{33})}$$

$$\boxed{\theta_6 = \text{atan2}(s_1 r_{12} - c_1 r_{22}, -s_1 r_{11} + c_1 r_{21})}$$

Kinematic Decoupling: IK solution for the Elbow Manipulator

$$\theta_1 = \text{atan2}(y_c, x_c) \quad \text{OR} \quad \theta_1 = \text{atan2}(y_c, x_c) + \pi$$

$$\theta_3 = \text{atan2}(\pm\sqrt{1 - D^2}, D)$$

$$\theta_2 = \text{atan2}(s, r) - \text{atan2}(a_3 sin\theta_3, a_2 + a_3 cos\theta_3)$$

$$\theta_5 = \text{atan2}(\pm\sqrt{1 - (s_1 r_{13} - c_1 r_{23})^2}, s_1 r_{13} - c_1 r_{23})$$

$$\theta_4 = \text{atan2}(-c_1 s_{23} r_{13} - s_1 s_{23} r_{23} + c_{23} r_{33}, c_1 c_{23} r_{13} + s_1 c_{23} r_{23} + s_{23} r_{33})$$

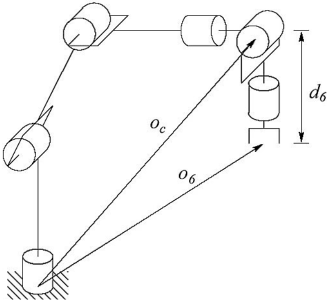
$$\theta_6 = \text{atan2}(s_1 r_{12} - c_1 r_{22}, -s_1 r_{11} + c_1 r_{21})$$

where,

$$r^2 = x_c^2 + y_c^2; \qquad s^2 = (z_c - d_1)^2$$

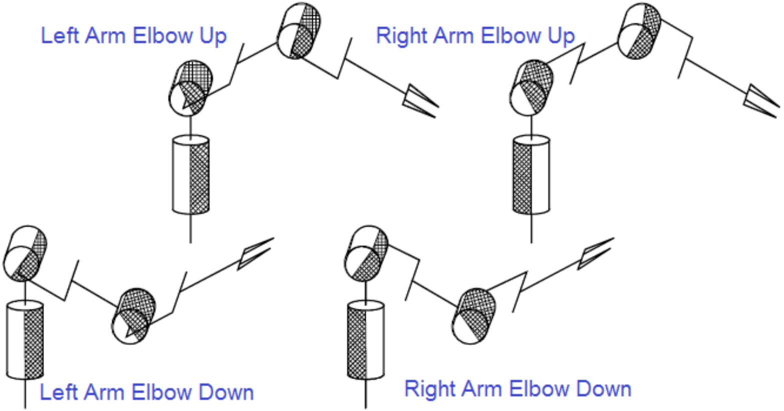
$$D = \frac{s^2 + r^2 - a_2^2 - a_3^2}{2a_2 a_3}$$

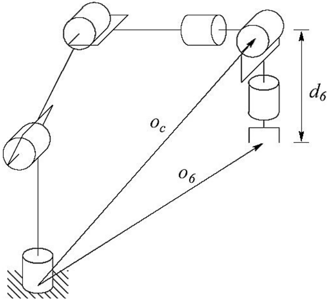
Four Solutions exist for the O_c!



Elbow Manipulator with Spherical Wrist

Kinematic Decoupling: IK solution for the Elbow Manipulator





Elbow Manipulator with Spherical Wrist

End of Lecture

haitham.elhussieny@gmail.com