Isentropic flow through Nozzles

- Conservation of momentum for a inviscid and steady flow

$$dp + \frac{1}{2}\rho d(V^2) + \gamma \cancelto{0}{dz} = 0$$

$$\frac{dp}{\rho V^2} = -\frac{dV}{V}$$

- conservation of mass

$$\dot{m} = \rho A V = \text{const.}$$

$$\text{Since } \dot{m} = \rho A V = c, \quad \therefore \ln \rho + \ln A + \ln V = c'$$

$$\text{differentiation} \rightarrow \frac{d\rho}{\rho} + \frac{dA}{A} + \frac{dV}{V} = 0$$

$$\frac{dp}{\rho V^2} = -\frac{dV}{V}$$

$$\rightarrow \quad -\frac{dV}{V} = \frac{d\rho}{\rho} + \frac{dA}{A} \quad \left(= \frac{dp}{\rho V^2}\right)$$

$$\frac{dA}{A}=\frac{dp}{\rho V^2}-\frac{d\rho}{\rho}=\frac{dp}{\rho V^2}(1-\frac{d\rho}{\rho}\cdot\frac{\rho V^2}{dp})$$

$$=\frac{dp}{\rho V^2}(1-\frac{V^2}{dp/d\rho})$$

Since $c=\sqrt{\left(\frac{\partial p}{\partial \rho}\right)_s}$, and $\mathrm{Ma}=\frac{V}{c}$

$$\therefore \frac{dp}{\rho V^2}(1-\mathrm{Ma}^2)=\frac{dA}{A}$$

$$\frac{dp}{\rho V^2}=-\frac{dV}{V}$$

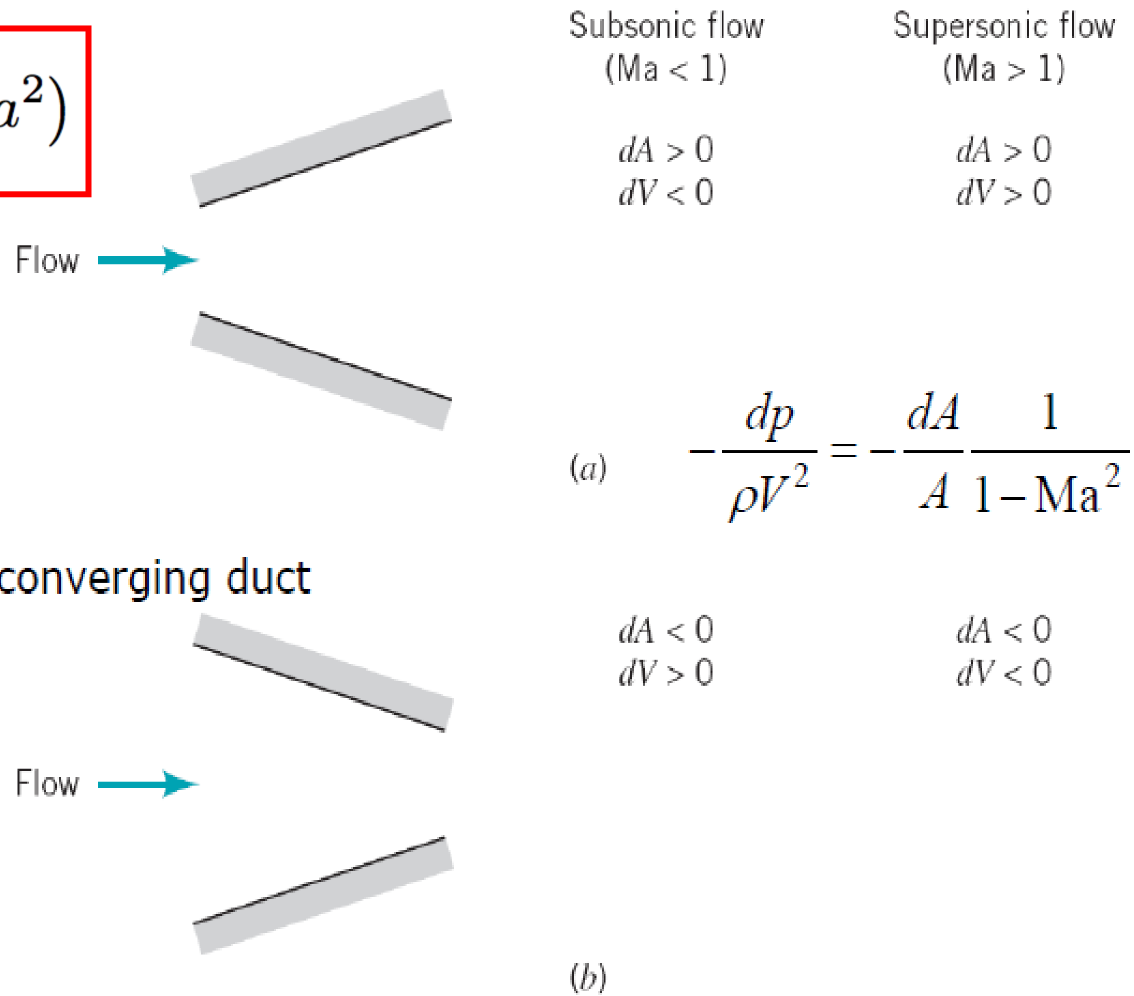
$$-\frac{dp}{\rho V^2}=-\frac{dA}{A}\frac{1}{1-\mathrm{Ma}^2}=\frac{dV}{V}$$

$$-\frac{dp}{\rho V^2} = -\frac{dA}{A}\frac{1}{1-\mathrm{Ma}^2} = \frac{dV}{V}$$

diverging duct

$$\boxed{\frac{dA}{A} = \frac{dP}{\rho V^2}\left(1 - Ma^2\right)}$$

converging duct



One-Dimensional Isentropic Flow
Variation of Fluid Velocity with Flow Area

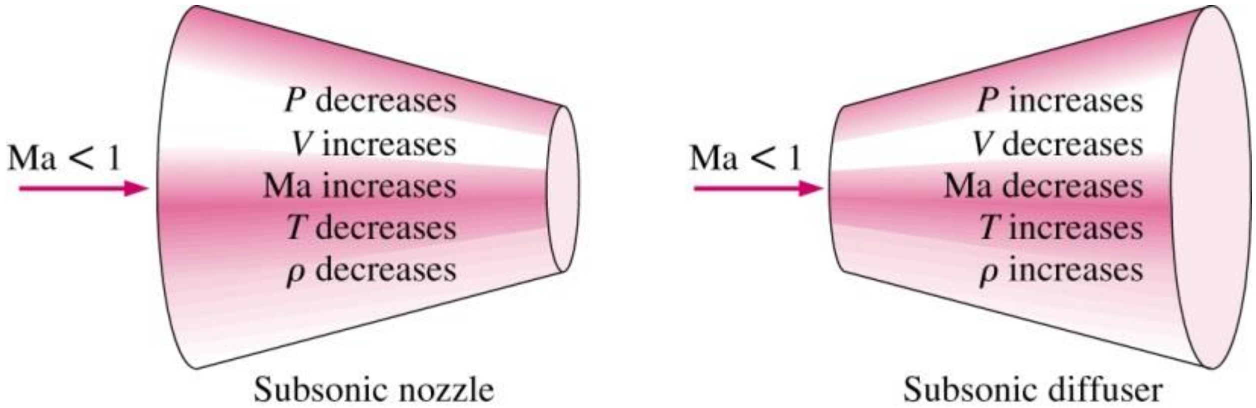
$$\boxed{\frac{dA}{A} = \frac{dP}{\rho V^2}\left(1 - Ma^2\right)}$$

$$-\frac{dp}{\rho V^2} = -\frac{dA}{A}\frac{1}{1-\mathrm{Ma}^2} = \frac{dV}{V}$$

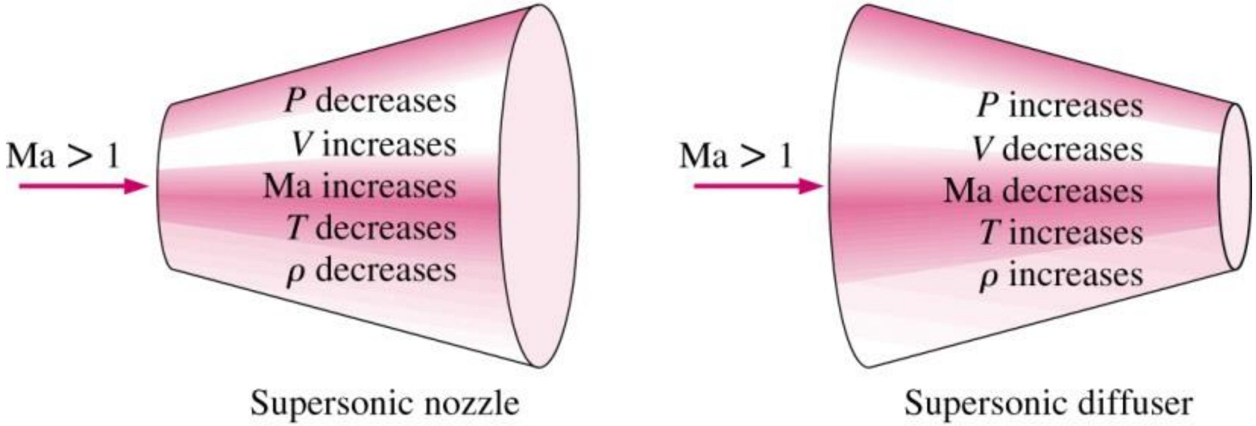
- This is an important relationship
  - For $Ma < 1$, $(1 - Ma^2)$ is positive $\Rightarrow$

  dA and *dP* have the same sign and *dV* has opposite sign.
    - Pressure of fluid must increase as the flow area of the duct increases, and must decrease as the flow area decreases
  - For $Ma > 1$, $(1 - Ma^2)$ is negative $\Rightarrow$ *dA* and *dP* have opposite signs and *dV* has same sign.
    - Pressure must increase as the flow area decreases, and must decrease as the area increases

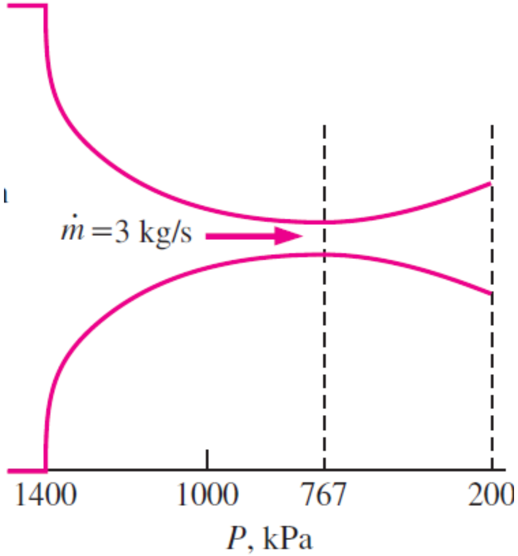
Comparison of flow properties in subsonic and supersonic nozzles and diffusers



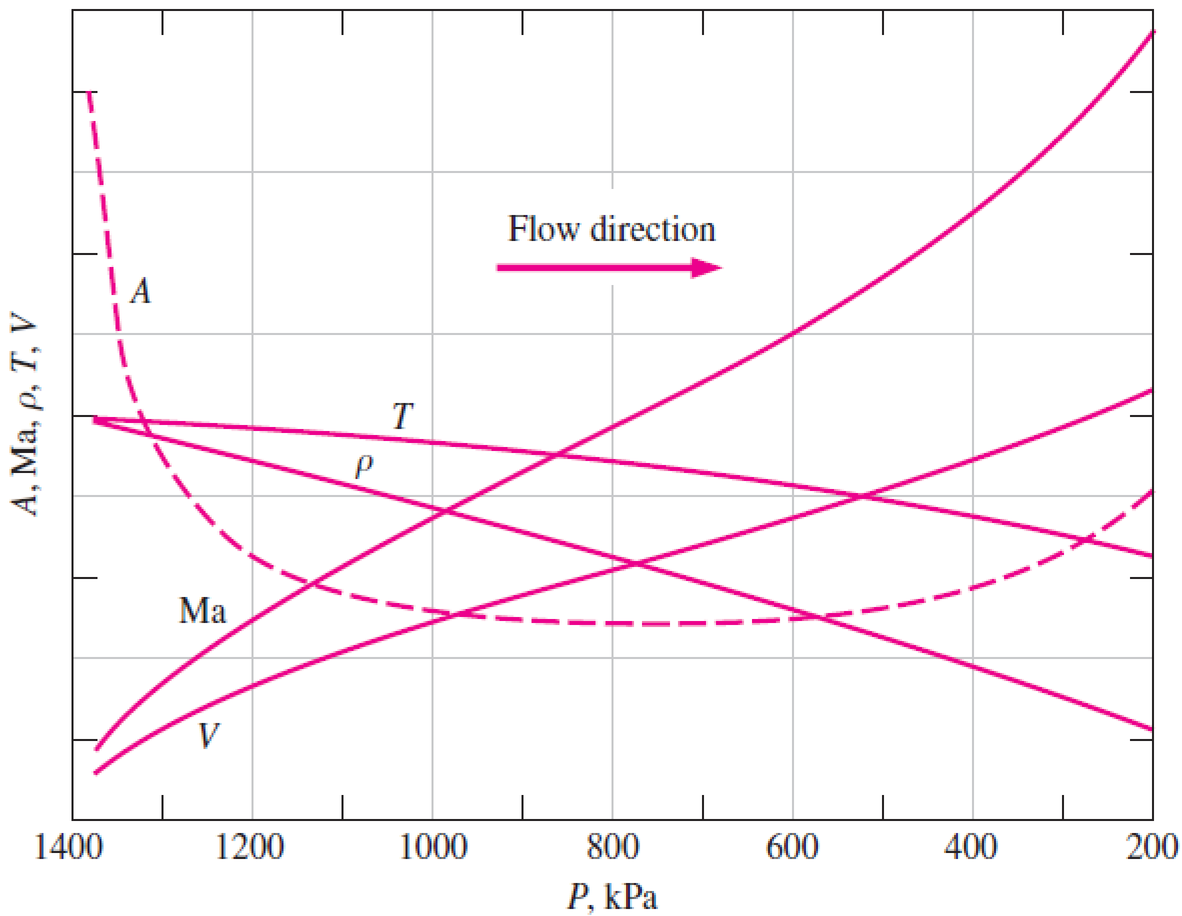
(*a*) Subsonic flow



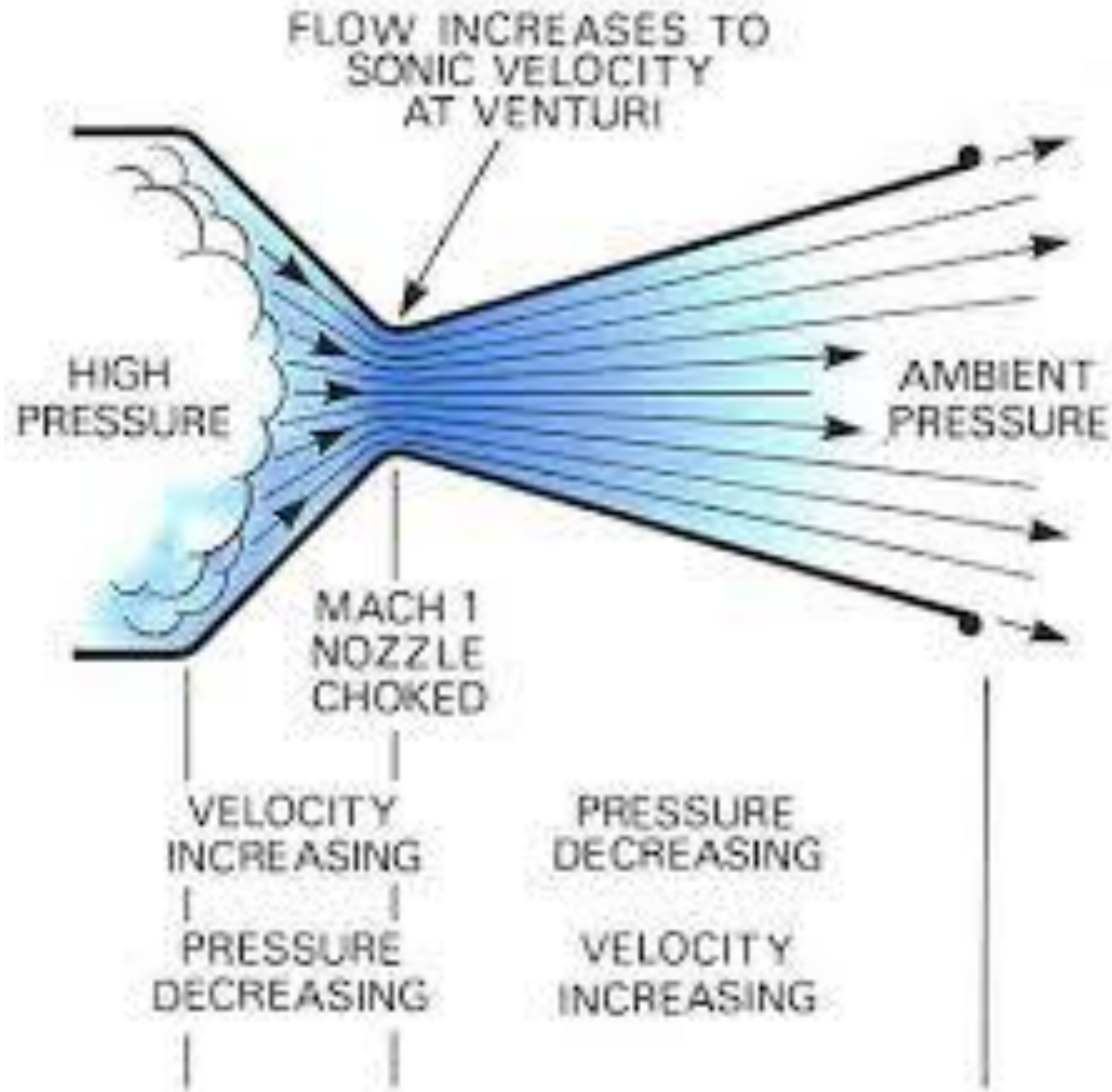
(*b*) Supersonic flow



$\dot{m}$ = 3 kg/s
1400
1000
767
200
P, kPa



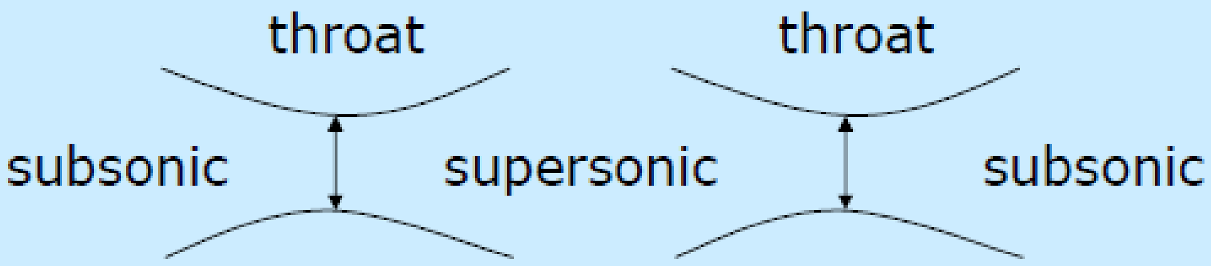
A, Ma, ρ, T, V
Flow direction
A
T
ρ
Ma
V
1400
1200
1000
800
600
400
200
P, kPa



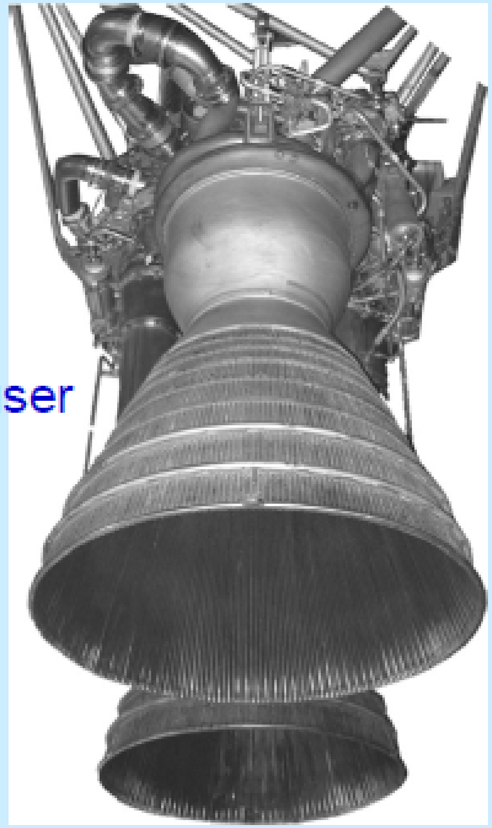
FLOW INCREASES TO
SONIC VELOCITY
AT VENTURI
HIGH
PRESSURE
AMBIENT
PRESSURE
MACH 1
NOZZLE
CHOKED
VELOCITY
INCREASING
PRESSURE
DECREASING
PRESSURE
DECREASING
VELOCITY
INCREASING

Therefore the sonic conduction Ma=1 can be obtained in a converging-diverging duct at the minimum area location.

For subsonic flow $\rightarrow$ converging diverging nozzle
For supersonic flow $\rightarrow$ converging diverging diffuser



Converging-diverging nozzle   Converging-diverging diffuser



Ratio *A/A**

$$\rho A V = \rho^* A^* V^* \quad \text{or} \quad \frac{A}{A^*} = \frac{\rho^*}{\rho}\frac{V^*}{V}$$

$$\text{with } V^* = \sqrt{kRT^*} \quad \text{and} \quad V = \mathrm{Ma}\sqrt{kRT}$$

$$\rightarrow \frac{A}{A^*} = \frac{\rho^*}{\rho}\frac{\sqrt{kRT^*}}{\sqrt{kRT}\,\mathrm{Ma}} = \frac{1}{\mathrm{Ma}}\frac{\rho^*}{\rho_0}\sqrt{\frac{T^*/T_0}{T/T_0}}$$

$$= \frac{1}{\mathrm{Ma}}\left(\frac{2}{k+1}\right)^{\frac{1}{k-1}}\left[1+\frac{k+1}{2}\mathrm{Ma}^2\right]^{\frac{1}{k-1}}\left[\frac{2/k+1}{1/1+[(k-1)/2]\mathrm{Ma}^2}\right]^{\frac{1}{2}}$$

$$\Rightarrow \frac{A}{A^*} = \frac{1}{\mathrm{Ma}}\left[\frac{1+[(k-1)/2]\mathrm{Ma}^2}{1+[(k-1)/2]}\right]^{\frac{k+1}{2(k-1)}}$$

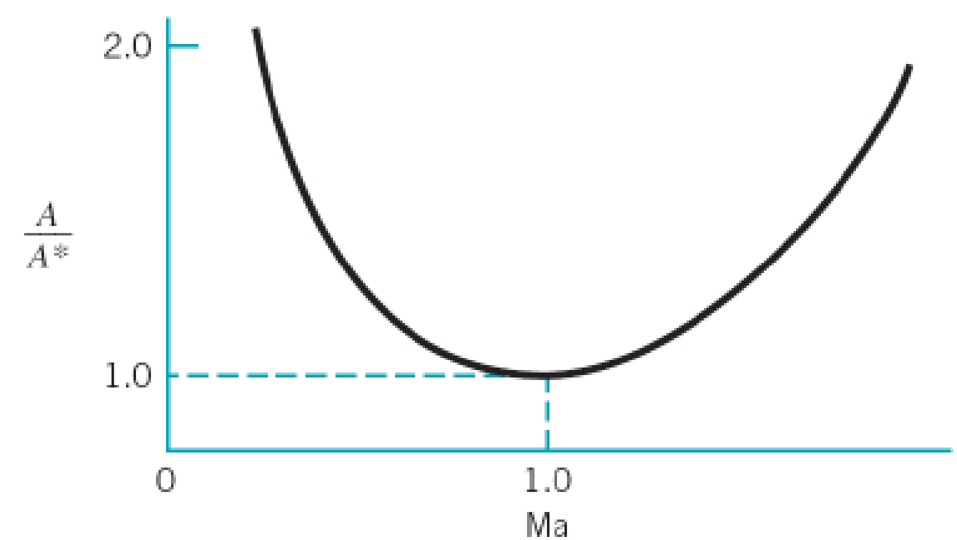
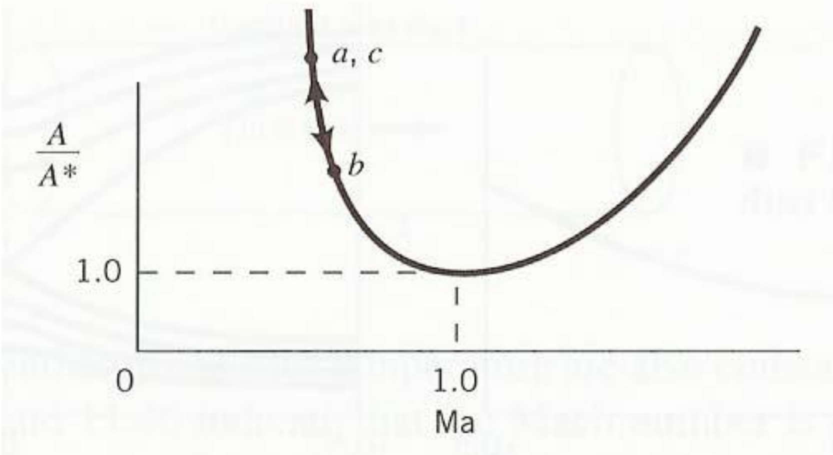
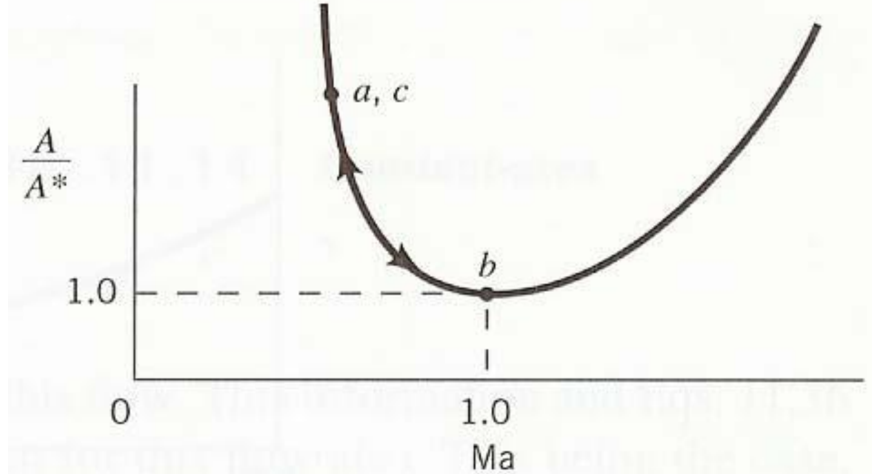


Figure 11.10 (p. 639)

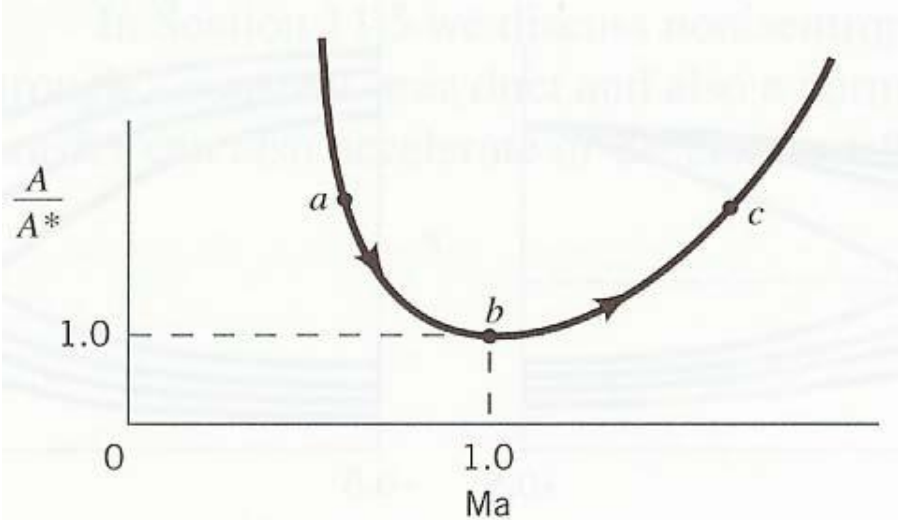
The variation of area ratio with Mach number for isentropic flow of an ideal gas (k = 1.4, linear coordinate scales).



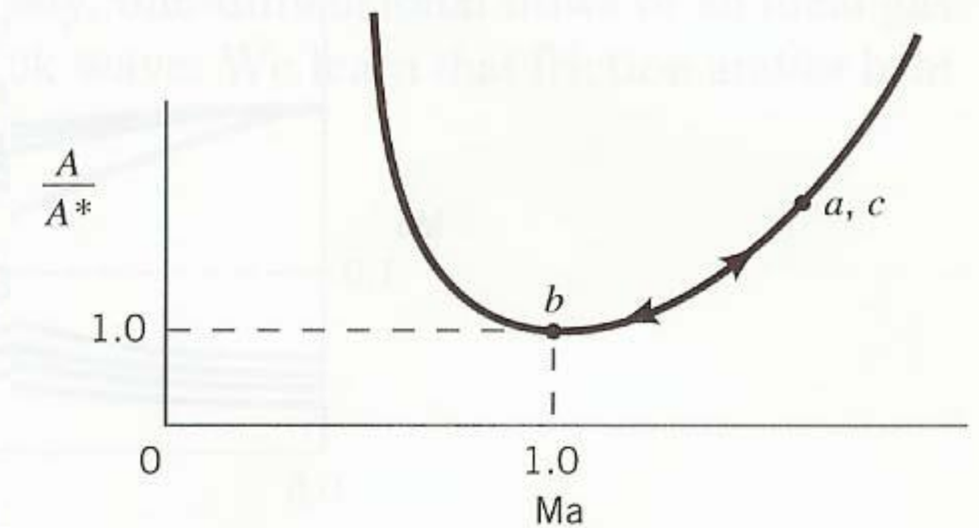
Subsonic -Subsonic



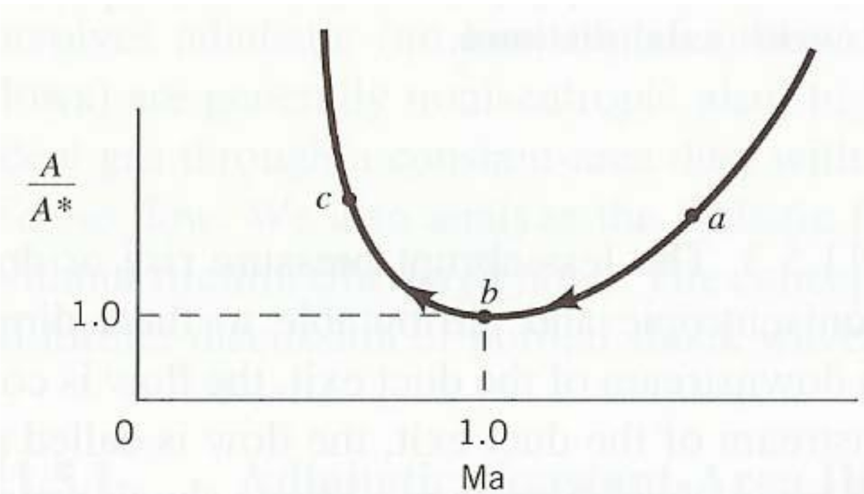
Subsonic- Subsonic (Chocked)



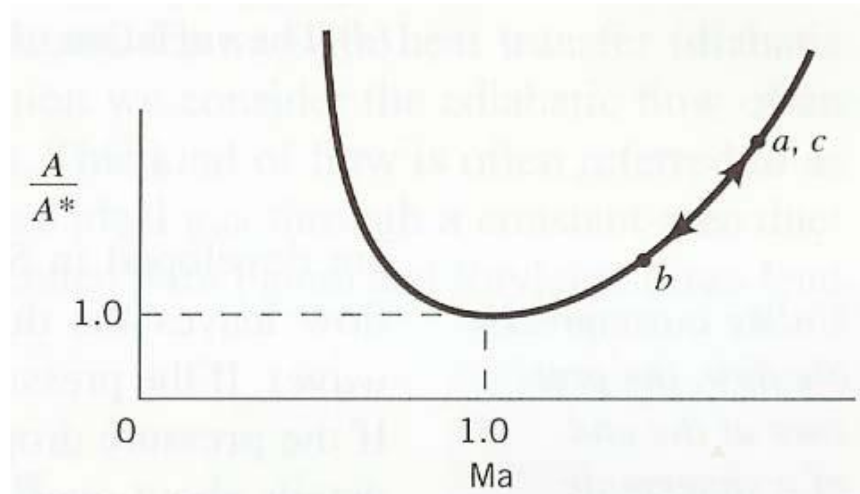
Subsonic – Supersonic (Chocked)



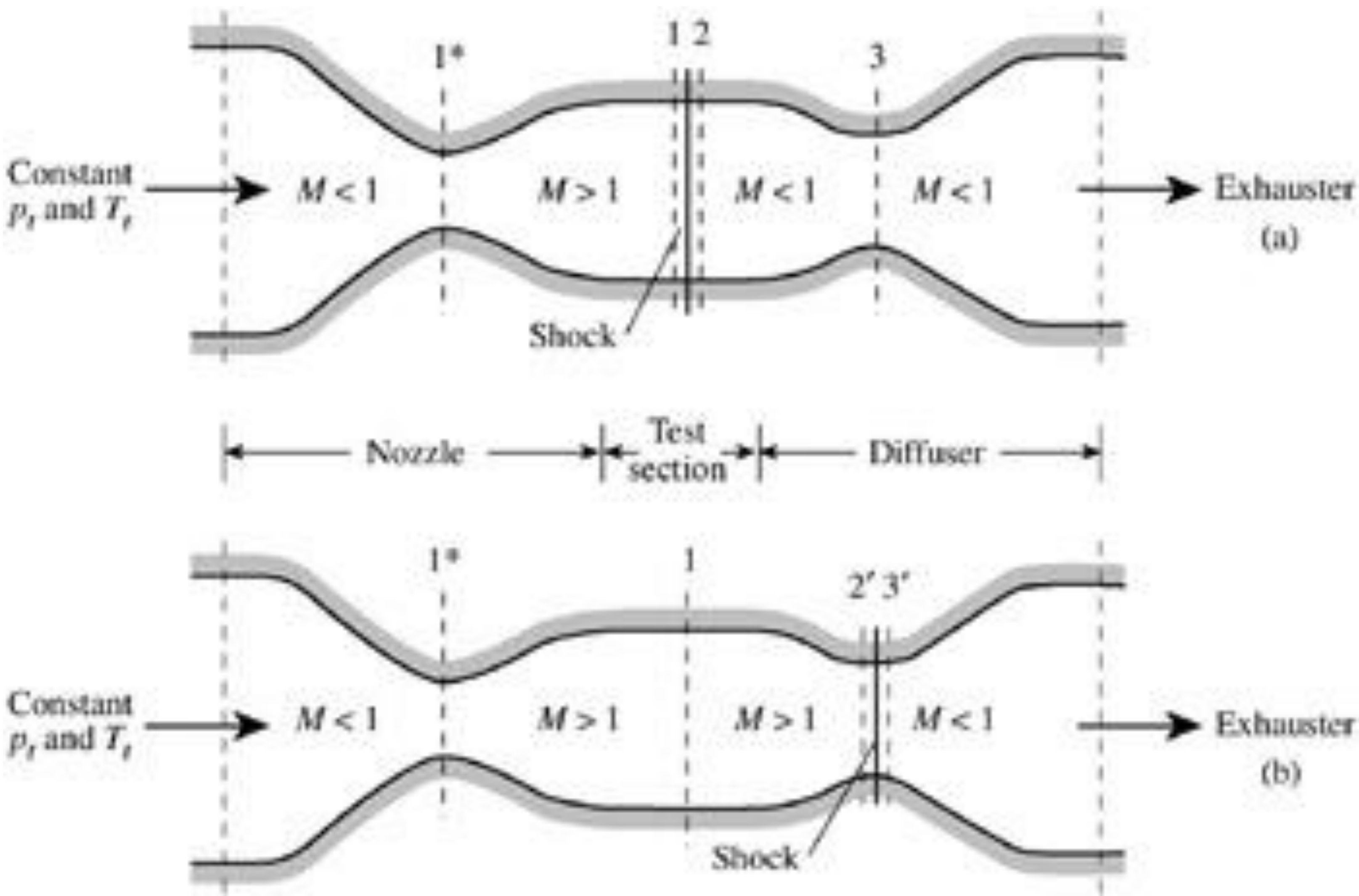
Supersonic-Supersonic (chocked)



Supersonic – Subsonic (Chocked)



Supersonic-supersonic (not-choked)



Constant
p_t and T_t
$M < 1$
1*
$M > 1$
1 2
$M < 1$
3
$M < 1$
Exhauster
(a)
Shock
Nozzle
Test section
Diffuser
Constant
p_t and T_t
$M < 1$
1*
$M > 1$
1
$M > 1$
2′ 3′
$M < 1$
Exhauster
(b)
Shock

Isentropic Flow Through Nozzles
Converging Nozzles

- Under steady flow conditions, mass flow rate is co

$$\dot{m} = \rho A V = \left(\frac{P}{RT}\right) A \left(Ma\sqrt{kRT}\right) = P A\, Ma \sqrt{\frac{k}{RT}}$$

- Substituting T and P from the expressions on slides 23 and 24 gives

$$\dot{m} = \frac{A\, Ma P_0 \sqrt{k/(RT_0)}}{\left[1 + (k-1)Ma^2/2\right]^{(k+1)/[2(k-1)]}}$$

  - Mass flow rate is a function of stagnation properties, flow area, and Ma

Isentropic Flow Through Nozzles
Converging Nozzles

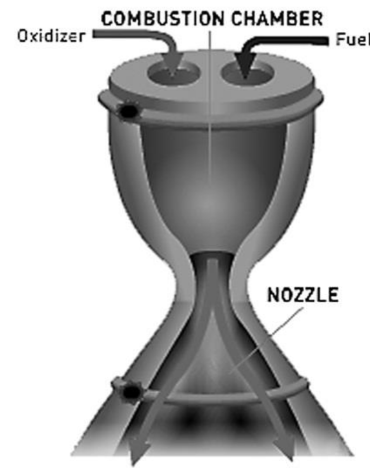
- The maximum mass flow rate through a nozzle with a given throat area A* is fixed by the P_0 and T_0 and occurs at Ma = 1

$$\dot{m} = A^* P_0 \sqrt{\frac{k}{RT_0}} \left(\frac{2}{k+1} \right)^{(k+1)/[2(k-1)]}$$

- This principal is important for chemical processes, medical devices, flow meters, and anywhere the mass flux of a gas must be known and controlled.

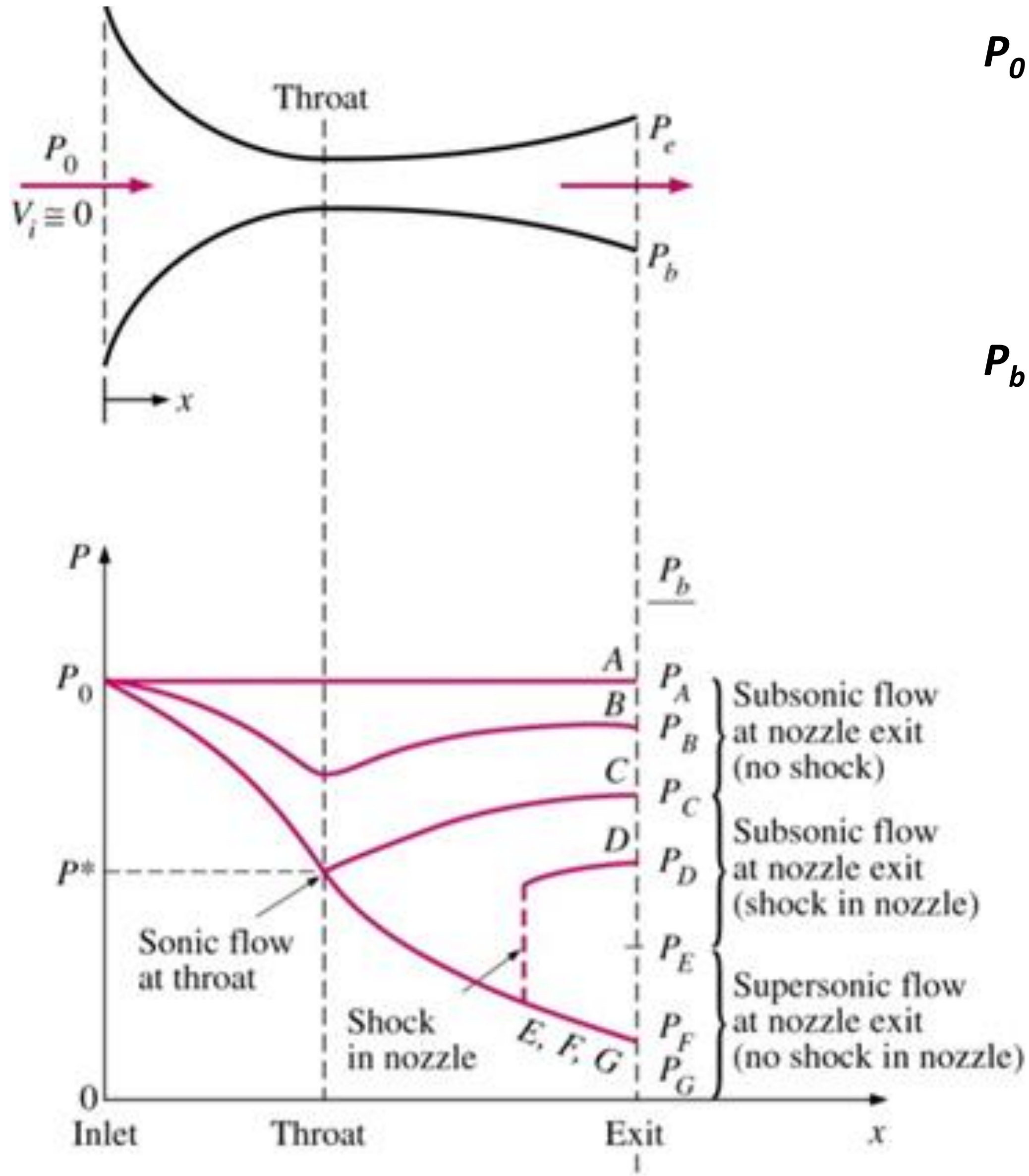
Isentropic Flow Through Nozzles
Converging-Diverging Nozzles

- The highest velocity in a converging nozzle is limited to the sonic velocity (Ma = 1), which occurs at the exit plane (throat) of the nozzle
- Accelerating a fluid to supersonic velocities (Ma > 1) requires a diverging flow section
  - Converging-diverging (C-D) nozzle
  - Standard equipment in supersonic aircraft and rocket propulsion
- Forcing fluid through a C-D nozzle does not guarantee supersonic velocity
  - Requires proper back pressure P_b



Isentropic Flow Through Nozzles

Converging-Diverging Nozzles

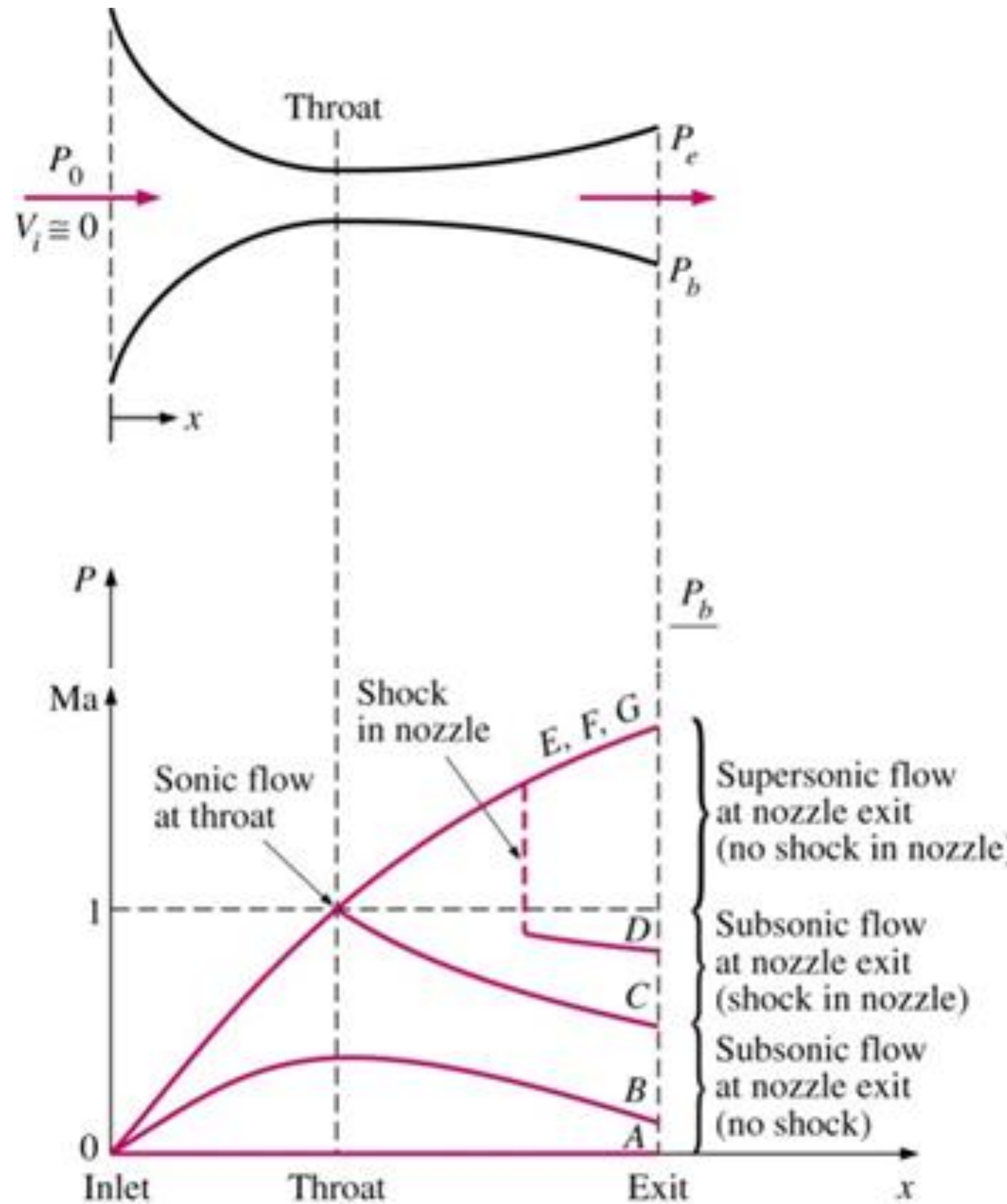


$P_0 > P_b > P_c$

– **Flow remains subsonic, and mass flow is less than for choked flow. Diverging section acts as diffuser**

$P_b = P_C$

– **Sonic flow achieved at throat. Diverging section acts as diffuser. Subsonic flow at exit. Further decrease in Pb has no effect on flow in converging portion of nozzle**



Throat
P_0
$V_i \cong 0$
P_e
P_b
x
P
P_b
Ma
Shock
in nozzle
E, F, G
Sonic flow
at throat
Supersonic flow
at nozzle exit
(no shock in nozzle)
1
D
Subsonic flow
at nozzle exit
(shock in nozzle)
C
Subsonic flow
at nozzle exit
(no shock)
B
A
0
Inlet
Throat
Exit
x


WIND TUNNEL