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NUMERICAL STUDY FOR THE FLOW AND HEAT TRANSFER IN A THIN LIQUID FILM OVER AN UNSTEADY STRETCHING SHEET WITH VARIABLE FLUID PROPERTIES IN THE PRESENCE OF THERMAL RADIATION

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ABSTRACT

In this paper, the effect of thermal radiation, variable viscosity and variable thermal conductivity on the flow and heat transfer of a thin liquid film over an unsteady stretching sheet is analyzed. The continuity, momentum and energy equations, which are coupled nonlinear partial differential equations, are reduced to a set of two non-linear ordinary differential equations, before being solved numerically. Results for the skin-friction coefficient, local Nusselt number, velocity profiles as well as temperature profiles are presented for different values of the governing parameters. It is found that increasing the viscosity parameter leads to a rise in the velocity near the surface of the sheet and a fall in the temperature. Furthermore, it is shown that the temperature increases due to an increase in the values of the thermal conductivity parameter and the thermal radiation parameter, while it decreases with an increase of the Prandtl number.

Keywords: Liquid film, Unsteady stretching sheet, Thermal radiation, Variable fluid properties, Similarity transformation.

1. INTRODUCTION

Fluid flow and heat transfer in a thin liquid film over a stretching sheet has gained considerable attention due to its many theoretical and technical applications in the engineering and technology fields. The knowledge of heat transfer within a thin liquid film is crucial in understanding the coating process and design of various heat exchangers and chemical processing equipments. Some applications include reactor fluidization, wire and fiber coating, polymer processing, food stuff processing, and transpiration cooling. Many metallurgical processes, such as drawing, annealing, and strips of filaments are done by drawing them through a quiescent fluid. The quality of the final product depends on the rate of heat transfer at the stretching surface. Crane [1] gave an exact similarity solution in closed analytical form for a steady two-dimensional boundary layer flow caused by the stretching of a flat sheet which moves in its own plane with velocity varying linearly with distance from a fixed point. Wang [2] was the first who studied the flow of a Newtonian fluid in a thin liquid film over an unsteady stretching sheet where he used a special type of transformation to express the boundary layer equations into their similarity form and also he solved the problem numerically and analytically. The axisymmetric motion of a fluid caused by an unsteady stretching surface has been investigated by Usha and Sridharan [3]. Later Andersson *et al*. [4] extended Wang's problem to the case of heat transfer. Dandapat *et al*. [5] investigated the effect of the thermocapillarity on the flow and heat transfer in a thin liquid film over an unsteady stretching sheet. Wang [6] presented exact analytical solutions for the momentum and heat transfer within a liquid film whose motion is caused solely by the unsteady stretching of a horizontal elastic sheet. The unsteady flow and heat transfer in a thin viscous liquid film over a heated horizontal stretching surface are analyzed by Santra and Dandapat [7]. The combined effect of viscous dissipation and magnetic field on the flow and heat transfer in a liquid film over an unsteady stretching surface was studied by Subhas Abel *et al*. [8]. Very recently, Noor and Hashim [9] investigated the effects of thermocapillarity and magnetic field in a thin liquid film on an unsteady elastic stretching sheet.

All of the above researchers [1-9] deal with the Newtonian fluids. Many materials such as polymer solutions or melts, drilling mud, certain oils, greases, pulps, and fossil fuels are classified as non-Newtonian

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fluids due to the non-linearity in the relationship between the stresses and the rates of the strain of these fluids. Andersson *et al*. [10] have studied the unsteady stretching flow in the case of finite thickness for a power-law fluid. Chen [11] studied the heat transfer occurring in a thin liquid film of a power-law fluid over an unsteady stretching sheet. The HAM solutions for the non-Newtonian problem considered by Andersson *et al*. [10] were presented by Wang and Pop [12]. Chen [13] investigated the effect of viscous dissipation on heat transfer in a power-law liquid film over an unsteady stretching surface. Siddiqui *et al*. [14] studied the thin film flow of non-Newtonian fluid on a vertically moving belt. Hayat *et al*. [15] investigated MHD flow and heat transfer of a second grade fluid film over an unsteady stretching sheet. Siddiqui *et al*. [16] studied the thin film flow problem with a third grade fluid and they obtained the solutions using the traditional perturbation method and the homotopy perturbation method. Hayat *et al*. [17] presented exact solutions for the problem of the thin film flow for a third grade fluid on an inclined plane.

All the studies mentioned above are assumed that the fluid has constant properties. Particularly, the physical property changes significantly with temperature. Therefore it is necessary to take the variation of viscosity and thermal conductivity into consideration. The viscosity is assumed to vary exponentially with the temperature [18,19] and the thermal conductivity is assumed to vary linearly with the temperature [20-22]. Dandapat *et al*. [23] discussed the effects of variable viscosity, variable thermal conductivity and thermocapillarity on the flow and heat transfer within a thin liquid film over an unsteady stretching sheet. Recently, Mahmoud and Megahed [24] studied the effects of variable viscosity and variable thermal conductivity on the flow and heat transfer of an electrically conducting non-Newtonian power-law fluid within a thin liquid film over an unsteady stretching sheet in the presence of a transverse magnetic field. The purpose of the present work is to study the effects of variable viscosity and variable thermal conductivity on the flow and heat transfer of a thin liquid film flow over an unsteady stretching sheet in the presence of a thermal radiation.

2. MATHEMATICAL FORMULATION

Consider the flow in a thin Newtonian liquid film of a uniform thickness $h(t)$ on a horizontal elastic sheet, which emerges from a narrow slit at the origin as shown in Fig. 1. The x-axis is chosen along the plane of the sheet and the y-axis is taken normal to the plane. We assume that the surface starts stretching from rest with the velocity $U(x, t)$ and temperature distribution $T_s(x, t)$.

The liquid film is assumed to be non-volatile and thin so that evaporation and buoyancy effect can be neglected. Further we assume that the viscosity and thermal conductivity vary with the temperature. The velocity and temperature fields of the thin liquid film obey the following boundary layer equations

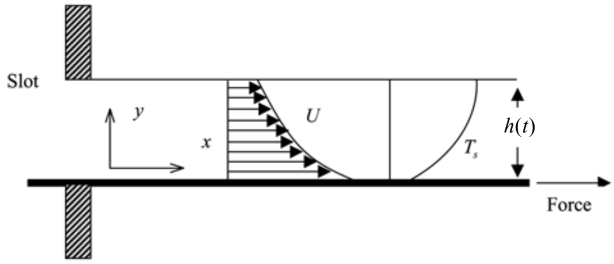

Fig. 1 Schematic of the physical system

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0\,, \tag{1}$$

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=\frac{1}{\rho}\frac{\partial}{\partial y}\left(\mu\frac{\partial u}{\partial y}\right), \tag{2}$$

$$\frac{\partial T}{\partial t}+u\frac{\partial T}{\partial x}+v\frac{\partial T}{\partial y}=\frac{1}{\rho c_p}\frac{\partial}{\partial y}\left(\kappa\frac{\partial T}{\partial y}\right)-\frac{1}{\rho c_p}\left(\frac{\partial q_r}{\partial y}\right), \tag{3}$$

where u and v are the velocity components along the x and y directions, respectively. t is the time, ρ is the fluid density, μ is the dynamic viscosity, T is the temperature of the fluid, κ is the thermal conductivity, q_r is the radiative heat flux, and c_p is the specific heat at constant pressure. It is noted that the heat conduction and thermal radiated terms, on the right hand side of Eq. (3), in "one-dimensional form" bear the assumption that the boundary layer approximation is valid. The appropriate boundary conditions for the present problem are:

$$u=U\,,\ v=0\,,\ T=T_s \quad \text{at} \quad y=0 \tag{4}$$

$$\frac{\partial u}{\partial y}=\frac{\partial T}{\partial y}=0 \quad \text{at} \quad y=h \tag{5}$$

$$v=\frac{dh}{dt} \quad \text{at} \quad y=h \tag{6}$$

where $h(t)$ is the thickness of the liquid film. The surface of the planar fluid is assumed to be smooth and free from surface waves. The influence of the interfacial shear due to quiescent gas is negligible, and therefore Eq. (5) describes a balance between viscous shear stress and net surface tension. The heat flux vanishes at the adiabatic free surface $y = h$. U is the surface velocity of the stretching sheet and the flow is caused by stretching the elastic surface at $y = 0$ such that the continuous sheet moves in the x-direction with the velocity:

$$U=\frac{bx}{(1-at)}\,, \tag{7}$$

where a and b are positive constants with dimension $(\text{time})^{-1}$. The temperature of the surface of the elastic sheet T_s is assumed to vary both along the sheet and with time [9],

$$T_s = T_0 - T_{\mathrm{ref}}\left[\frac{dx^2}{2(\mu_0/\rho)}\right](1-at)^{-3/2}, \tag{8}$$

where d is a constant, T_0 is the temperature at the slit and T_{ref} is the reference temperature. The radiative heat flux q_r is employed according to Rosseland approximation [25] such that

$$q_r = -\frac{4\sigma^*}{3k^*}\frac{\partial T^4}{\partial y}, \tag{9}$$

where σ^* is the Stefan-Boltzmann constant and k^* is the mean absorption coefficient. Following Raptis [26], we assume that the temperature difference within the flow is small such that it may be expressed as a linear function of the temperature. Expanding T^4 in a Taylor series about T_0 and neglecting higher-order terms, we have:

$$T^4 \cong 4T_0^3 T - 3T_0^4. \tag{10}$$

The special form of the surface velocity (7) and the surface temperature (8) allows the system of partial differential Eqs. (2) and (3) to be transformed to a system of coupled nonlinear ordinary differential equations, using the following similarity transformations [9,12]:

$$\eta = \left(\frac{b}{\mu_0/\rho}\right)^{1/2}(1-at)^{-1/2}\beta^{-1}y, \tag{11}$$

$$u = bx(1-at)^{-1}f'(\eta), \tag{12}$$

$$v = -\left(\frac{\mu_0}{\rho}b\right)^{1/2}(1-at)^{-1/2}\beta f(\eta), \tag{13}$$

$$T = T_0 - T_{ref}\left[\frac{dx^2}{2(\mu_0/\rho)}\right](1-at)^{-3/2}\,\theta(\eta), \tag{14}$$

where f is the dimensionless stream function, θ is the dimensionless temperature of the fluid and β is yet an unknown constant denoting the dimensionless film thickness which can be defined as [9,12]:

$$\beta = \left(\frac{b}{\mu_0/\rho}\right)^{1/2}(1-at)^{-1/2}h(t). \tag{15}$$

The variation of the viscosity μ and the thermal conductivity κ with temperature is assumed to be in the following form [23,24]:

$$\mu = \mu_0 e^{\alpha\theta}, \tag{16}$$

$$\kappa = \kappa_0(1+\varepsilon\theta), \tag{17}$$

where μ_0 and κ_0 are the viscosity and thermal conductivity at the ambient, respectively. $\alpha = -\alpha'(T_s - T_0)$ is the viscosity parameter, α' is the positive fluid property and ε is the thermal conductivity parameter.

Using Eqs. (16) and (17), the mathematical problem defined in Eqs. (1) ~ (6) are then transformed into a set of ordinary differential equations and their associated boundary conditions:

$$e^{\alpha\theta}(f''' + \alpha\theta' f'') + \gamma\left[f f'' - f'^2 - S(f' + \frac{1}{2}\eta f'')\right] = 0, \tag{18}$$

$$\frac{1}{\Pr}\left[(1+R+\varepsilon\theta)\theta'' + \varepsilon\theta'^2\right] + \gamma\left[f\theta' - 2f'\theta - S\left(\frac{3}{2}\theta + \frac{1}{2}\eta\theta'\right)\right] = 0, \tag{19}$$

$$f = 0,\ f' = 1,\ \theta = 1 \ \text{at} \ \eta = 0, \tag{20}$$

$$f'' = 0,\ \ \theta' = 0 \ \text{at} \ \eta = 1, \tag{21}$$

$$f = \frac{S}{2} \ \text{at} \ \eta = 1, \tag{22}$$

where a prime denotes differentiation with respect to η, $S = a/b$ is the unsteadiness parameter, $\Pr = \mu_0 c_p / \kappa_0$ is the Prandtl number, $\gamma = \beta^2$ is the dimensionless film thickness, and $R = 16\sigma^* T_0^3 / 3k^*\kappa_0$ is the radiation parameter.

The physical quantities of interest are the skin-friction coefficient C_f and the local Nusselt number Nu_x which are defined as:

$$C_f = \frac{\tau_w}{\rho U^2/2}, \quad Nu_x = \frac{xq_w}{\kappa_0 T_{\mathrm{ref}}}. \tag{23}$$

Further, τ_w and q_w are the shear stress and the heat transfer from the surface of the plate, respectively, and they are given by:

$$\tau_w = -\mu_0\left(\frac{\partial u}{\partial y}\right)_{y=0}, \quad q_w = -\kappa_0\left(\frac{\partial T}{\partial y}\right)_{y=0}. \tag{24}$$

Using Eqs. (11) ~ (14), we obtain

$$C_f = \frac{-2}{\beta}f''(0)\,\mathrm{Re}_x^{-1/2}, \tag{25}$$

$$Nu_x = \frac{1}{2\beta(1-at)^{1/2}}\frac{d}{b}\theta'(0)\,\mathrm{Re}_x^{3/2}, \tag{26}$$

where $\mathrm{Re}_x = \rho U x / \mu_0$ is the local Reynolds number.

3. NUMERICAL SOLUTION

The coupled system of non-linear ordinary differential Eqs. (18) and (19) together with the boundary conditions (20) ~ (22) is solved numerically using the most efficient numerical shooting technique with the fourth-order Runge-Kutta scheme. The estimated value of γ is therefore systematically adjusted until Eq. (22) is satisfied within 10^{-7}. Once the convergence is achieved, the resulting differential equations can be integrated using fourth-order Runge-Kutta integration scheme. The above procedure is repeated until we get the results up to the desired degree of accuracy, 10^{-5}. To assess the accuracy of the present method, results for our problem in the absence of radiation Parameter (R = 0), constant fluid viscosity (α = 0) and constant fluid thermal conductivity (ε = 0) are compared with those obtained by Wang and Pop [12] (in the case of Newtonian fluid) and Noor and Hashim [9] (when there are no effects for thermocapillarity and magnetic field) as shown in Table 1. Good agreement is observed from that Table.

4. RESULTS AND DISCUSSION

For the problem of thin liquid film flow, there exist specific values for the unsteadiness parameter S in which no solution could be obtained. Wang [2] found that (for positive value of S in a Newtonian fluid) the dimensionless film thickness parameter is a monotonically decreasing function of S within the interval [0,2] and no solution exists outside this interval. The limiting case of $S \to 0$ stands for the case of an infinitely thick layer of fluid, *i.e.* $\beta \to \infty$, whereas the limiting case of $S \to 2$ represents a liquid film of infinitesimal thickness, *i.e.* $\beta \to 0$. Variations of the dimensionless film thickness β, skin-friction coefficient in terms of $-f''(0)$, free surface temperature $\theta(1)$, and local Nusselt number in terms of $-\theta'(0)$ for the parameters governing the flow and heat transfer are presented in Tables 2 ~ 6. From Table 2, we can observe that increasing the measure of the unsteadiness parameter S will decrease the thin film thickness $\beta = \gamma^{1/2}$, the values of the skin-friction coefficient $-f''(0)$ and the heat flux $-\theta'(0)$, but increase the free surface temperature $\theta(1)$. Figure 2 illustrates the effect of the unsteadiness parameter on the velocity profile. The results show that the velocity increases along the surface with an increase in the unsteadiness parameter. As the unsteadiness parameter increases, the same behaviour is observed for the temperature $\theta(\eta)$ in Fig. 3. When the unsteady stretching parameter S on the surface rises, friction and the flow velocity also rise. When friction increases, the area of the stretching surface in contact with the flow increases, therefore heat generated from the friction on the surface is transferred to the flow. This leads to a rise in the surface temperature and the flow is heated. Table 3 shows that, as the viscosity parameter α increases, the film thickness $\beta = \gamma^{1/2}$ and the heat flux $-\theta'(0)$ also increase. This physically leads to faster cooling of the thin film flow, which is important in many engineering applications. Correspondingly, the free surface temperature $\theta(1)$ and the skin-friction coefficient $-f''(0)$ decrease. Figure 4 illustrates the effect of viscosity parameter α on the velocity profiles. It can be shown that the velocity increases near the surface with an increase in the viscosity parameter but the reverse is true away from the sheet. But the temperature $\theta(\eta)$ decreases as the viscosity parameter increases as shown from Fig. 5. It is obvious from Table 4 that increasing in the Prandtl number Pr leads to an increase for the

Table 1 Comparison for values of $f''(0)$ and γ

S	Noor and Hashim [9]		Wang and Pop [12]		Present work	
	γ	$f''(0)$	γ	$f''(0)$	γ	$f''(0)$
1.4	0.674089	−1.01278	0.674097	−1.01278	0.674093	−1.012779
1.6	0.331976	−0.64240	0.331977	−0.64241	0.331975	−0.642396
1.8	0.127013	−0.309138	0.127014	−0.309138	0.127012	−0.309135

Table 2 Variation of $\beta = \gamma^{1/2}$, $f''(0)$, $\theta(1)$ and $\theta'(0)$ when $\alpha = 0.1$, $\varepsilon = 0.1$, $R = 1$ and Pr = 1

S	β	$-f''(0)$	$\theta(1)$	$-\theta'(0)$
0.8	2.226702	2.60647	0.233955	2.41802
1.0	1.601771	1.92514	0.348752	1.77790
1.2	1.173380	1.41414	0.473161	1.29644
1.4	0.856434	0.99741	0.602940	0.90581
1.6	0.602561	0.63577	0.735144	0.57114

Table 3 Variation of $\beta = \gamma^{1/2}$, $f''(0)$, $\theta(1)$ and $\theta'(0)$ when $S = 0.8$, $\varepsilon = 0.1$, $R = 1$ and Pr = 1

α	β	$-f''(0)$	$\theta(1)$	$-\theta'(0)$
0.0	2.15199	2.68095	0.251616	2.31637
0.1	2.22670	2.60647	0.233955	2.41802
0.2	2.30130	2.52864	0.217410	2.51963
0.4	2.44959	2.36492	0.187610	2.72202

Table 4 Variation of $\beta = \gamma^{1/2}$, $f''(0)$, $\theta(1)$ and $\theta'(0)$ when $\alpha = 0.1$, $\varepsilon = 0.1$, $R = 1$ and $S = 0.8$

Pr	β	$-f''(0)$	$\theta(1)$	$-\theta'(0)$
0.7	2.23242	2.61862	0.331022	1.93869
1.0	2.22670	2.60647	0.233955	2.41802
2.0	2.21520	2.58173	0.092074	3.59682
3.0	2.20855	2.56710	0.043626	4.48395

Table 5 Variation of $\beta = \gamma^{1/2}$, $f''(0)$, $\theta(1)$ and $\theta'(0)$ when $\alpha = 0.1$, Pr = 1, $R = 1$ and $S = 0.8$

ε	β	$-f''(0)$	$\theta(1)$	$-\theta'(0)$
0.0	2.22598	2.60493	0.227308	2.50253
0.1	2.22670	2.60647	0.233955	2.41802
0.2	2.22733	2.60788	0.240646	2.34034
0.6	2.22978	2.61313	0.267616	2.08341

Table 6 Variation of $\beta = \gamma^{1/2}$, $f''(0)$, $\theta(1)$ and $\theta'(0)$ when $\alpha = 0.1$, Pr = 1, $\varepsilon = 0.1$ and $S = 0.8$

R	β	$-f''(0)$	$\theta(1)$	$-\theta'(0)$
0.0	2.21585	2.58320	0.0950347	3.48459
1.0	2.22670	2.60647	0.233955	2.41802
2.0	2.23302	2.61982	0.342149	1.90080
3.0	2.23728	2.62880	0.424915	1.57986

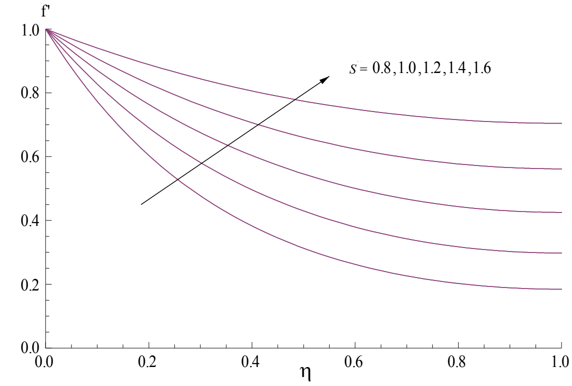

Fig. 2 Velocity profiles for various values of S with $\varepsilon = \alpha = 0.1$, $R = 1$ and Pr = 1

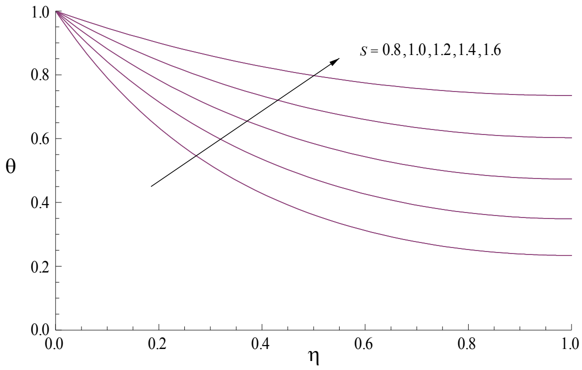

Fig. 3 Temperature profiles for various values of S with $\varepsilon = \alpha = 0.1$, $R = 1$ and Pr = 1

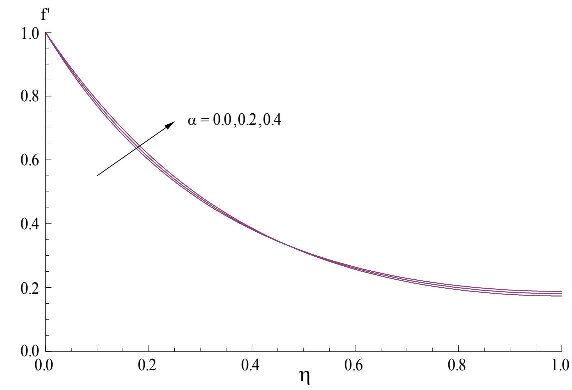

Fig. 4 Velocity profiles for various values of α with $\varepsilon = 0.1$, $S = 0.8$, $R = 1$ and Pr = 1

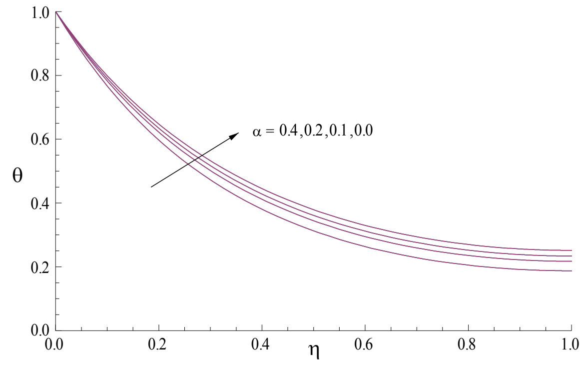

Fig. 5 Temperature profiles for various values of α with $\varepsilon = 0.1$, $S = 0.8$, $R = 1$ and Pr = 1

heat flux $-\theta'(0)$, whereas the free surface temperature $\theta(1)$, skin-friction coefficient $-f''(0)$ and the thin film thickness $\beta = \gamma^{1/2}$ are found to be decreased. The effect of the Prandtl number on the dimensionless temperature is illustrated in Fig. 6. From this figure it is found that the temperature decreases with increasing the Prandtl number (*i.e.* decreasing thermal diffusivity), which reflects to the usual reduction of temperature distribution owing to the effect of increasing Pr. This causes a raise in the rate of heat transfer between the flow and surface, and speeds up the cooling of the thin film flow. As seen from Tables 5 and 6, one can observe that increasing the thermal conductivity parameter ε and the radiation parameter R causes an enhancement for values of skin-friction coefficient $-f''(0)$, thin film thickness and the free surface temperature but the reverse is true for the values of heat flux $-\theta'(0)$. The effects of the thermal conductivity parameter ε and the thermal radiation parameter on the temperature profile θ are presented in Figs. 7 and 8, respectively. From these figures it can be seen that the temperature distribution increases as the thermal conductivity parameter and the thermal radiation parameter increase, which leads to a fall in the rate of heat transfer $-\theta'(0)$ from the flow to the surface. This in turn causes a fall in the rate of cooling for the liquid film. Furthermore, Fig. 9 reveals that, for great values of thermal conductivity parameter ε and viscosity parameter α, the velocity decreases along the surface with increase in the thermal radiation parameter but the reverse is true away from the sheet.

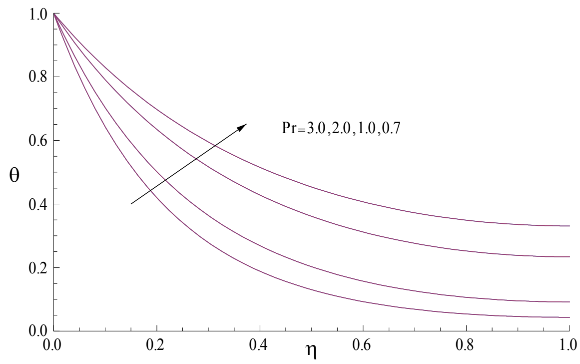

Fig. 6 Temperature profiles for various values of Pr with $\varepsilon = 0.1$, $S = 0.8$, $R = 1$ and $\alpha = 0.1$

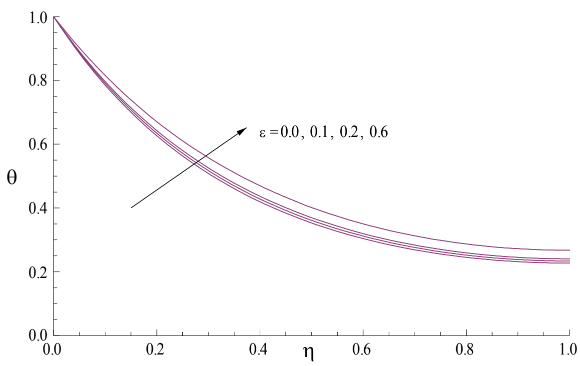

Fig. 7 Temperature profiles for various values of ε with Pr = 1, $S = 0.8$, $R = 1$ and $\alpha = 0.1$

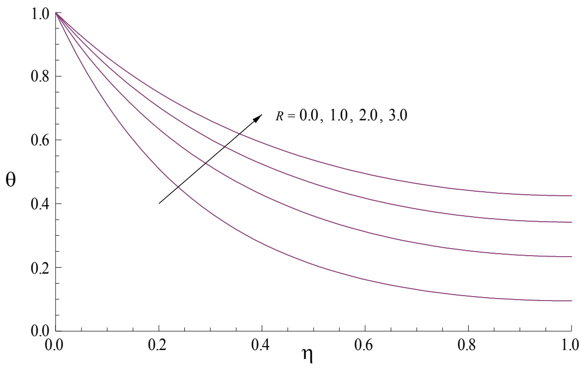

Fig. 8 Temperature profiles for various values of R with Pr = 1, S = 0.8, ε = 0.1 and α = 0.1

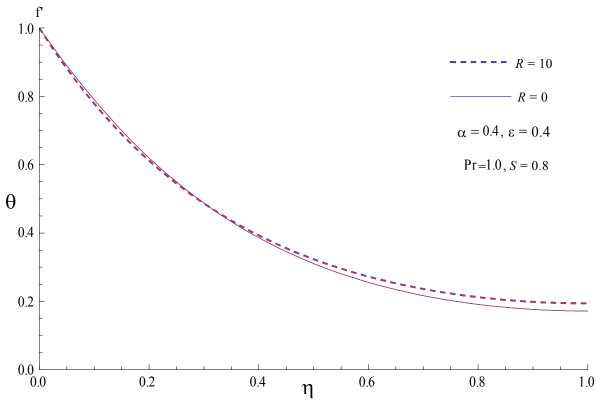

Fig. 9 Velocity profiles for various values of R

5. CONCLUSIONS

The problem of flow and heat transfer of a thin liquid film over an unsteady stretching sheet in the presence of thermal radiation is studied in the light of variation of fluid properties with the temperature. The obtained similarity ordinary differential equations are solved numerically by using the fourth-order Runge-Kutta scheme coupled with the shooting technique. In this study there exhibits a very important role for the variation of viscosity and thermal conductivity with temperature on velocity field, temperature distribution, the free surface temperature, skin-friction coefficient and heat flux. It is found that increasing the unsteadiness parameter causes a rise in the flow velocity and temperature. On the other hand, the thin film thickness and the heat flux decrease with increasing the unsteadiness parameter. Furthermore, as the viscosity parameter, thermal conductivity parameter and the thermal radiation parameter increase the thin film thickness increases.

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