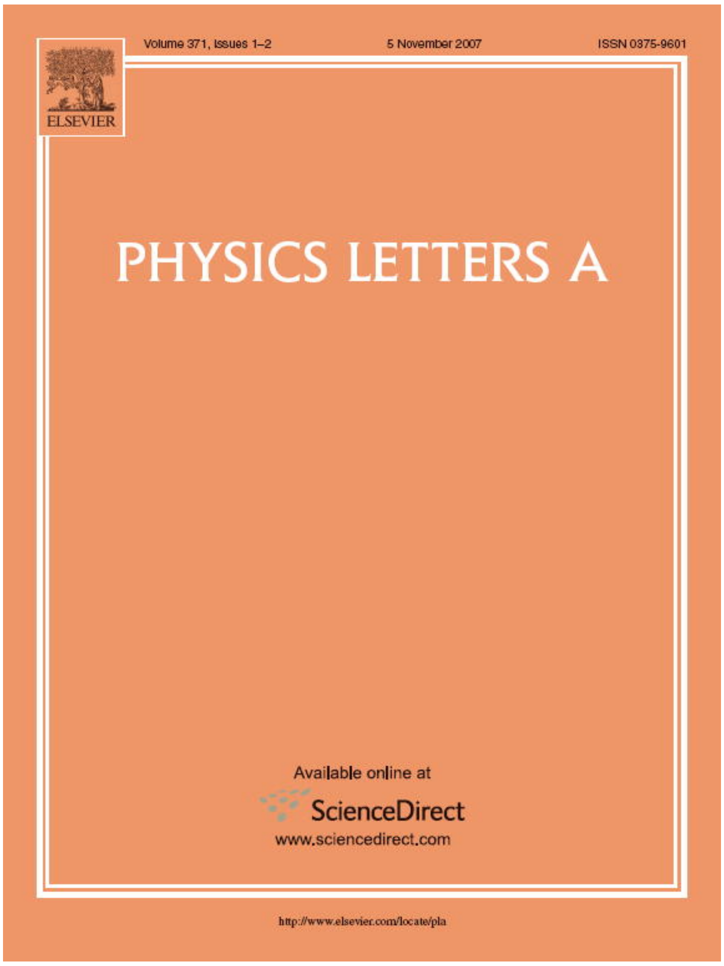
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Numerical studies for a multi-order fractional differential equation

N.H. Sweilam [a,*], M.M. Khader [b], R.F. Al-Bar [c]

[a] *Department of Mathematics, Faculty of Science, Cairo University, Giza, Egypt*
[b] *Department of Mathematics, Faculty of Science, Benha University, Benha, Egypt*
[c] *Department of Mathematics, Faculty of Science, Umm Al-Qura University, Saudi Arabia*

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Abstract

In this Letter, we implement the variational iteration method and the homotopy perturbation method, for solving the system of fraction differential equations (FDE) generated by a multi-order fraction differential equation. The fractional derivatives are described in the Caputo sense. In these schemes, the solution takes the form of a convergent series with easily computable components. Numerical results show that the two approaches are easy to implement and accurate when applied to partial differential equations of fractional order. An algorithm to convert a multi-order FDE has been suggested which is valid in the most general cases.

Keywords: Variational iteration method; Homotopy perturbation method; Lagrange multiplier; Fractional differential equation; Caputo fractional derivative

1. Introduction

In recent years a lot of attention has been devoted to study the variational iteration method (VIM) and the homotopy perturbation method (HPM) given by J.H. He (see [9–11] and the references sited therein), for solving numerically a wide range of problems whose mathematical models yield differential equation or system of differential equations (see also [1,2,24,25], and the references therein). The main reasons for the success of these methods are no need to discretization of the variables, and no requirement of large computer memory. Many authors (see [1,2,9–11,24], and the references cited therein) are pointed out that the VIM and HPM can overcome the difficulties arising in calculation of Adomian's polynomials in Adomian's decomposition method (see [4–6] and the references therein).

Fractional differential equations (FDE) have been of considerable interest in the recent literature [6,7,12,17,20–23]. This topic has received a great deal of attention especially in the fields of viscoelastic materials [3,15,16,23], electrochemical processes [13], dielectric polarization [26], colored noise [27], anomalous diffusion, signal processing [19], control theory [22], advection and dispersion of solutes in natural porous or fractured media [4,5] and chaos [18]. Djrbashian and Nersesian [8] considered the Cauchy problem with multi-term fractional derivatives, and proved that the Cauchy problem has a unique solution. Kilbas et al. [14] gave a solution of Volterra integro-differential equations with generalized Mittag–Leffler function in the kernels. The main reason for the success of the theory in these cases is that these new fractional-order models are more accurate than integer-order models.

For nonlinear FDE, however, one mainly resorts to numerical methods (see [6,7] and the references sited therein). These numerical methods involve discretization of the variables, which gives rise to rounding off errors and the requirement of large computer memory. Another drawback of numerical methods stems [6] are the difficulties arising in calculation of Adomian's polynomials in Adomian's decomposition method (see [1,2,9–11,24], and the references cited therein).

* Corresponding author.
E-mail address: n_sweilam@yahoo.com (N.H. Sweilam).

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In the present work we investigate the multi-order FDE. Analytical questions of existence and uniqueness of solutions have been discussed in the literature [7] and the references sited therein.

2. Basic definitions

We give some basic definitions and properties of the fractional calculus theory which are used further in this Letter.

Definition 2.1. A real function $f(x), x > 0$, is said to be in the space $C_\mu, \mu \in R$ if there exists a real number $p(> \mu)$, such that $f(x) = x^p f_1(x)$, where $f_1(x) \in [0, \infty)$, and it is said to be in the space C_μ^m if and only if $f^{(m)} \in C_\mu, m \in N$.

Definition 2.2. The Riemann–Liouville fractional integral operator of order $\alpha > 0$, of a function $f \in C_\mu$, $\mu \geqslant -1$, is defined as

$$J^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} f(t)\,dt, \quad \alpha > 0,\ x > 0, \quad J^0 f(x) = f(x). \tag{1}$$

Properties of the operator J^α can be found in [20–22]. We mention only the following. For $f \in C_\mu, \mu \geqslant -1, \alpha, \beta \geqslant 0, \gamma \geqslant -1$
(1) $J^\alpha J^\beta f(x) = J^{\alpha+\beta} f(x)$,
(2) $J^\alpha J^\beta f(x) = J^\beta J^\alpha f(x)$,
(3) $J^\alpha x^\gamma = \frac{\Gamma(\gamma+1)}{\Gamma(\alpha+\gamma+1)} x^{\alpha+\gamma}$.

The Riemann–Liouville derivative has certain disadvantages when trying to model real-world phenomena with fractional differential equations. Therefore, we shall introduce a modified fractional differential operator D_*^α proposed by Caputo in his work on the theory of viscoelasticity [14].

Definition 2.3. The fractional derivative of $f(x)$ in the Caputo sense is defined as

$$D_*^\alpha f(x) = J^{m-\alpha} D^m f(x) = \frac{1}{\Gamma(m-\alpha)} \int_0^x (x-t)^{m-\alpha-1} f^{(m)}(t)\,dt, \quad \alpha > 0,\ x > 0, \tag{2}$$

for $m - 1 < \alpha \leqslant m$, $m \in N$, $x > 0$, $f \in C_{-1}^m$.

Lemma 2.1. *If $m - 1 < \alpha \leqslant m$, $m \in N$, and $f \in C_\mu^m$, $\mu \geqslant -1$, then $D_*^\alpha J^\alpha f(x) = f(x)$, and*

$$J^\alpha D_*^\alpha f(x) = f(x) - \sum_{k=0}^{m-1} f^{(k)}(0^+) \frac{x^k}{k!}, \quad x > 0. \tag{3}$$

Lemma 2.2. *(See [7].) Let $y(t) \in C^k[0, T > 0]$ for some $T > 0$ and $k \in N$ and let $q \notin N$ be such that $0 < q < k$. Then $D_*^q y(0) = 0$.*

3. Multi-order FDE as a system of FDE

In this section, we present an algorithm to convert the multi-order fractional differential equation into a system of FDE. Consider the following multi-order FDE:

$$D_*^\alpha y(t) = F\big(t, y(t), D_*^{\beta_1} y(t), \ldots, D_*^{\beta_n} y(t)\big), \quad y^{(k)}(0) = c_k, \quad k = 0, \ldots, m, \tag{4}$$

where $m < \alpha \leqslant m + 1$, $0 < \beta_1 < \beta_2 < \cdots < \alpha$ and D_*^α denotes Caputo fractional derivative of order α. It should be noted that F can be nonlinear in general. Eq. (4) can be represented as a system of FDE, as follows,

$$\text{Set } y_1 = y \text{ and define } D_*^{\beta_1} y_1 = y_2. \tag{5}$$

We will, now consider the following cases:

Case 1. If $m - 1 \leqslant \beta_1 < \beta_2 \leqslant m$ then define

$$D_*^{\beta_2-\beta_1} y_2 = y_3. \tag{6}$$

Claim. $y_3 = D_*^{\beta_2} y$. *If $\beta_1 = m - 1$, then $D_*^{\beta_2-\beta_1} y_2 = D_*^{\beta_2-(m-1)} y^{(m-1)} = D_*^{\beta_2} y_1$.*

Hence the claim. If $\beta_1 \notin N$, then by Lemma 2.2, $D_*^{\beta_1} y_1(0) = 0$ and as $\beta_2 - \beta_1 < 1$,

$$D_*^{\beta_2-\beta_1}\big[D_*^{\beta_1} y_1\big] = D^{\beta_2-\beta_1}\big[D_*^{\beta_1} y_1\big] = DJ^{1+\beta_2-\beta_1} J^{m-\beta_1} y_1^{(m)} = DJ^{1+m-\beta_2} y_1^{(m)} = J^{m-\beta_2} y_1^{(m)} = D_*^{\beta_2} y_1 = D_*^{\beta_2} y, \tag{7}$$

therefore $y_3 = D_*^{\beta_2-\beta_1} y_2 = D_*^{\beta_2} y$.

Case 2. Consider $m-1 \leqslant \beta_1 < m \leqslant \beta_2$ $\beta_1 = m-1$, then define $D_*^{\beta_2-\beta_1} y_2 = y_3$,

$$D_*^{\beta_2-\beta_1} y_2 = D_*^{\beta_2-m+1} y_1^{(m-1)} = D_*^{\beta_2} y_1.$$

$$\text{If } m-1 < \beta_1 < m \leqslant \beta_2, \text{ then define } D_*^{m-\beta_1} y_2 = y_3. \tag{8}$$

Claim. $y_3 = y^{(m)}$. *As $\beta_1 \notin N$, $D_*^{\beta_1} y_1(0) = y_2(0) = 0$ (in view of Lemma 2.2), and $0 < m - \beta_1 < 1$,*

$$D_*^{m-\beta_1} y_2 = D^{m-\beta_1} y_2 = DJ^{1+\beta_1-m} J^{m-\beta_1} y_1^{(m)} = DJ y_1^{(m)} = y_1^{(m)} = y^{(m)}. \tag{9}$$

Hence $y_3 = y^{(m)}$. Further define:

$$D_*^{\beta_2-m} y_3 = y_4. \tag{10}$$

Claim. $y_4 = D_*^{\beta_2} y$. *As $y_4 = D_*^{\beta_2-m} y_3 = D_*^{\beta_2-m} y^{(m)} = D_*^{\beta_2} y$.*

Continuing similarly we can convert the initial value problem (4) into a system of FDE.

The following example will illustrate the method. Consider

$$D_*^{3.6} y(t) = F\big(t, y(t), D_*^{1.2} y, D_*^{1.7} y, D_*^{2.1} y, D_*^{3.5} y\big),$$

where $y(0) = c_0$, $y'(0) = c_1$, $y''(0) = c_2$ and $y'''(0) = c_3$. This initial value problem can be viewed as the following system of FDE.

$$\begin{aligned}
&D_*^{1.2} y_1(t) = y_2(t), \quad y_1(0) = c_0, \quad y_1'(0) = c_1,\\
&D_*^{0.5} y_2(t) = y_3(t)\big[= D_*^{1.7} y(t)\big], \quad y_2(0) = 0,\\
&D_*^{0.3} y_3(t) = y_4(t)\big[= y''(t)\big], \quad y_3(0) = 0,\\
&D_*^{0.1} y_4(t) = y_5(t)\big[= D_*^{2.1} y(t)\big], \quad y_4(0) = c_2,\\
&D_*^{0.9} y_5(t) = y_6(t)\big[= y'''(t)\big], \quad y_5(0) = 0,\\
&D_*^{0.5} y_6(t) = y_7(t)\big[= D_*^{3.5} y(t)\big], \quad y_6(0) = c_3,\\
&D_*^{0.1} y_7(t) = F(t, y_1, y_2, y_3, y_5, y_7), \quad y_7(0) = 0,
\end{aligned}$$

where $y_1(t) = y(t)$.

4. Varitional iteration method and a system of FDE

In this section, we present the analysis of the Varitional iteration method, and apply it with some illustrative examples.

4.1. Analysis of VIM

To illustrate the basic concepts of variational iteration method, consider the multi-order Eq. (4) as system of fractional differential equations:

$$D_*^{\alpha_i} y_i(t) = y_{i+1}(t), \quad i = 1, 2, \ldots, n-1, \tag{11}$$

$$D_*^{\alpha_n} y_n(t) = F(t, y_1, y_2, \ldots, y_n), \tag{12}$$

with initial conditions are:

$$y_i^k(0) = c_k^i, \quad 0 \leqslant k \leqslant m_i, \quad m_i < \alpha_i \leqslant m_{i+1}, \quad 1 \leqslant i \leqslant n.$$

According to the variational iteration method, we can construct the following iteration formula:

$$y_{i,p+1}(t) = y_{i,p}(t) + \int_0^t \lambda_{i,1}(\tau)\left[\frac{\partial^{m_i}}{\partial \tau^{m_i}}\big(y_{i,p}(\tau)\big) - \hat{y}_{i+1,p}(\tau)\right] d\tau, \quad i = 1(1)n-1, \tag{13}$$

$$y_{n,p+1}(t)=y_{n,p}(t)+\int_0^t \lambda_{n,2}(\tau)\left[\frac{\partial^{m_n}}{\partial\tau^{m_n}}\big(y_{n,p}(\tau)\big)-F(t,\hat{y}_{1,p},\hat{y}_{2,p},\ldots,\hat{y}_{n,p})\right]d\tau, \tag{14}$$

where $\lambda_{i,1}$, $i=1(1)n-1$, and $\lambda_{n,2}$ are general Lagrange multipliers, which can be identified optimally via variational theory. The second term on the right-hand side in (13) and (14) are called the correction and the subscript p denotes the pth order approximation. Under a suitable restricted variational assumptions (i.e. $\hat{y}_{i,p}$ are considered as a restricted variation), we can assume that the above correctional functional are stationary (i.e. $\delta\hat{y}_{i,p}=0$), then the Lagrange multipliers can be identified. Now we can start with the given initial approximation and by the above iteration formulas we can obtain the approximate solutions.

In an algorithmic form, the VIM can be expressed and implemented the solutions as follows:

Algorithm. Let p be the iteration index, set a suitable value for the tolerance (Tol.)
Step 0: Choose a suitable $y_{1,0}(t)$ and $y_{2,0}(t)$.
Step 1: Set $p=0$.
Step 2: Use the calculated values of $y_{1,p}(t)$ and $y_{2,p}(t)$ to compute $y_{1,p+1}(t)$ from Eq. (21).
Step 3: Define $y_{1,p}:=y_{1,p+1}$.
Step 4: Use the calculated values of $y_{1,p}(t)$ and $y_{2,p}(t)$ to compute $y_{2,p+1}(t)$ from Eq. (22).
Step 5: Define $y_{2,p}:=y_{2,p+1}$.
Step 6: If $\max|y_{1,p}-y_{1,p-1}|<\text{Tol}$ and $\max|y_{2,p}-y_{2,p-1}|<\text{Tol}$ stop, otherwise continue.
Step 7: Define $y_{1,p+1}:=y_{1,p}$.
Step 8: Set $p=p+1$, and return to step 2.

4.2. Illustrative examples by VIM

Example 1. Consider the following initial value problem in case of the inhomogeneous Bagley–Torvik equation [7]:

$$D_*^2y(t)+D_*^{1.5}y(t)+y(t)=1+t,\qquad y(0)=1,\qquad y'(0)=1. \tag{15a}$$

In view of the discussion in Section 3, Eqs. (15) can be viewed as the following system of FDE:

$$\begin{aligned}&D_*^{1.5}y_1(t)=y_2(t),\qquad y_1(0)=y'_1(0)=1,\\&D_*^{0.5}y_2(t)=-y_2(t)-y_1(t)+1+t,\qquad y_2(0)=0.\end{aligned} \tag{15b}$$

According to the variational iteration method, we can construct the following iteration formula:

$$y_{1,p+1}(t)=y_{1,p}(t)+\int_0^t \lambda_{1,1}(\tau)\left[\frac{\partial^2}{\partial\tau^2}\big(y_{1,p}(\tau)\big)-\hat{y}_{2,p}(\tau)\right]d\tau, \tag{16}$$

$$y_{2,p+1}(t)=y_{2,p}(t)+\int_0^t \lambda_{2,2}(\tau)\left[\frac{\partial}{\partial\tau}\big(y_{2,p}(\tau)\big)+\hat{y}_{2,p}+\hat{y}_{1,p}-1-\tau\right]d\tau. \tag{17}$$

Calculating variation with respect to $y_{1,p}$, $y_{2,p}$ respectively as follows:

$$\delta y_{1,p+1}(t)=\delta y_{1,p}(t)+\delta\int_0^t \lambda_{1,1}(\tau)\left[\frac{\partial^2}{\partial\tau^2}\big(y_{1,p}(\tau)\big)-\hat{y}_{2,p}(\tau)\right]d\tau, \tag{a1}$$

$$\delta y_{1,p+1}(t)=\delta y_{1,p}(t)+\delta\int_0^t \lambda_{1,1}(\tau)\left[\frac{\partial^2 y_{1,p}}{\partial\tau^2}\right]d\tau, \tag{b1}$$

$$\delta y_{1,p+1}=\delta y_{1,p}-\delta y_{1,p}\lambda'_{1,1}(\tau)|_{\tau=t}+\delta y'_{1,p}\lambda_{1,1}(\tau)|_{\tau=t}+\int_0^t \delta y_{1,p}\left[\frac{\partial^2\lambda_{1,1}}{\partial\tau^2}\right]d\tau, \tag{c1}$$

$$\delta y_{2,p+1}(t)=\delta y_{2,p}(t)+\delta\int_0^t \lambda_{2,2}(\tau)\left[\frac{\partial}{\partial\tau}\big(y_{2,p}(\tau)\big)+\hat{y}_{2,p}+\hat{y}_{1,p}-1-\tau\right]d\tau, \tag{a2}$$

$$\delta y_{2,p+1}(t)=\delta y_{2,p}(t)+\delta\int_0^t \lambda_{2,2}(\tau)\left[\frac{\partial y_{2,p}}{\partial\tau}\right]d\tau, \tag{b2}$$

$$\delta y_{2,p+1} = \delta y_{2,p} + \delta y_{2,p}\lambda'_{2,2}|_{\tau=t} + \int_0^t \delta y_{2,p}\left[\frac{\partial\lambda_{2,2}}{\partial\tau}\right]d\tau. \tag{c2}$$

Consequently, the following stationary conditions are obtained:

$$\lambda''_{1,1}(\tau) = 0, \qquad \lambda_{1,1}(\tau)|_{\tau=t} = 0, \qquad 1 - \lambda'_{1,1}(\tau)|_{\tau=t} = 0, \tag{18}$$
$$\lambda'_{2,2}(\tau) = 0, \qquad 1 + \lambda_{2,2}(\tau)|_{\tau=t} = 0. \tag{19}$$

The Lagrange multipliers, therefore, can be identified as

$$\lambda_{1,1}(\tau) = \tau - t, \qquad \lambda_{2,2}(\tau) = -1. \tag{20}$$

Substituting the identified multiplier into Eqs. (16) and (17) results the following iteration formula:

$$y_{1,p+1}(t) = y_{1,p}(t) + \int_0^t (\tau - t)\left[\frac{\partial^{1.5}}{\partial\tau^{1.5}}\big(y_{1,p}(\tau)\big) - y_{2,p}(\tau)\right]d\tau, \tag{21}$$

$$y_{2,p+1}(t) = y_{2,p}(t) - \int_0^t \left[\frac{\partial^{0.5}}{\partial\tau^{0.5}}\big(y_{2,p}(\tau)\big) + y_{2,p} + y_{1,p} - 1 - \tau\right]d\tau. \tag{22}$$

Eqs. (21) and (22) can be solved iteratively using $y_1(0) = y'_1(0) = 1$, $y_2(0) = 0$, as an initial approximation. We can start with the given initial approximation using initial conditions:

$$y_{1,0} = 1 + t, \qquad y_{2,0} = 0.$$

By using (21) and (22), some approximate solutions are listed below:

$$y_{1,1}(t) = 1 + t + \int_0^t (\tau - t)\left[\frac{\partial^{1.5}}{\partial\tau^{1.5}}(1 + t)\right]d\tau = 1 + t,$$

$$y_{2,1}(t) = -\int_0^t \left[\frac{\partial^{0.5}}{\partial\tau^{0.5}}(0) + (1 + \tau) - 1 - \tau\right]d\tau = 0.$$

And so on, $y_{1,p}(t) = 1 + t$, $p \geqslant 0$ and $y_{2,p}(t) = 0$, $p \geqslant 0$.

In view of the above terms, we find $y_1(t) = 1 + t$, and $y_2(t) = 0$. So $y(t) = 1 + t$ is the required solution of (15).

Example 2. Consider the following initial value problem

$$D_*^3 y(t) + D_*^{2.5} y(t) + y^2(t) = t^4, \quad y(0) = y'(0) = 0, \quad y''(0) = 2. \tag{23a}$$

In view of the discussion in Section 3, if we choose $y(t) = y_1$ and $D_*^{2.5} y(t) = y_2$ then Eq. (23a) can be viewed as the following system of FDE:

$$D_*^{2.5} y_1(t) = y_2(t), \quad y_1(0) = y'_1(0) = 0, \quad y''_1(0) = 2,$$
$$D_*^{0.5} y_2(t) = -y_2(t) - y_1^2(t) + t^4, \quad y_2(0) = 0. \tag{23b}$$

According to the variational iteration method, we can construct the following iteration formula:

$$y_{1,p+1}(t) = y_{1,p}(t) + \int_0^t \lambda_{1,1}(\tau)\left[\frac{\partial^3}{\partial\tau^3}\big(y_{1,p}(\tau)\big) - \hat{y}_{2,p}(\tau)\right]d\tau, \tag{24}$$

$$y_{2,p+1}(t) = y_{2,p}(t) + \int_0^t \lambda_{2,2}(\tau)\left[\frac{\partial}{\partial\tau}\big(y_{2,p}(\tau)\big) + \hat{y}_{2,p} + \hat{y}_{1,p}^2 - \tau^4\right]d\tau. \tag{25}$$

By the same way, the following stationary conditions are obtained:

$$\lambda'''_{1,1}(\tau) = 0, \qquad \lambda_{1,1}(\tau)|_{\tau=t} = 0, \qquad \lambda'_{1,1}(\tau)|_{\tau=t} = 0 \quad \text{and} \quad \lambda''_{1,1}(\tau)|_{\tau=t} = 0, \tag{26}$$
$$\lambda'_{2,2}(\tau) = 0, \qquad 1 + \lambda_{2,2}(\tau)|_{\tau=t} = 0. \tag{27}$$

The Lagrange multipliers, therefore, can be identified as

$$\lambda_{1,1}(\tau) = \frac{-1}{2}\tau^2 + t\tau - \frac{1}{2}t^2, \qquad \lambda_{2,2}(\tau) = -1. \tag{28}$$

Substituting the identified multiplier into Eqs. (24) and (25) results the following iteration formula:

$$y_{1,p+1}(t) = y_{1,p}(t) + \int_0^t \left(\frac{-1}{2}\tau^2 + t\tau - \frac{1}{2}t^2\right)\left[\frac{\partial^{2.5}}{\partial\tau^{2.5}}\big(y_{1,p}(\tau)\big) - y_{2,p}(\tau)\right]d\tau, \tag{29}$$

$$y_{2,p+1}(t) = y_{2,p}(t) - \int_0^t \left[\frac{\partial^{0.5}}{\partial\tau^{0.5}}\big(y_{2,p}(\tau)\big) + y_{2,p} + y_{1,p}^2 - \tau^4\right]d\tau, \quad p \geqslant 0. \tag{30}$$

The second term on the right is called the correction term. Eqs. (21) and (22) can be solved iteratively using $y_1(0) = y_1'(0) = 0$, $y_1''(0) = 2$, $y_2(0) = 0$ as an initial approximation. We can start with the given initial approximation using initial conditions:

$$y_{1,0} = t^2, \qquad y_{2,0} = 0.$$

By the formulas (29) and (30), some approximate solutions are listed below:

$$y_{1,1}(t) = t^2 + \int_0^t \left(\frac{-1}{2}\tau^2 + t\tau - \frac{1}{2}t^2\right)\left[\frac{\partial^{2.5}}{\partial\tau^{2.5}}\big(\tau^2\big)\right]d\tau = t^2,$$

$$y_{2,1}(t) = -\int_0^t \left[\frac{\partial^{0.5}}{\partial\tau^{0.5}}(0) + \tau^4 - \tau^4\right]d\tau = 0,$$

and so on, $y_{1,p}(t) = t^2$, $p \geqslant 0$ and $y_{2,p}(t) = 0$, $p \geqslant 0$.

In view of the above terms, we find $y_1(t) = t^2$, and $y_2(t) = 0$. So $y(t) = t^2$ is the required solution of (23a).

5. Homotopy perturbation method and a system of FDE

In this section, we present the analysis of the HPM, and apply it with some illustrative examples.

5.1. Analysis of HPM

To illustrate the basic concepts of HPM, consider the multi-order equation (4) as system of fractional differential equations (11) and (12)

$$D_*^{\alpha_i} y_i(t) = y_{i+1}(t), \quad i = 1, 2, \ldots, n-1,$$
$$D_*^{\alpha_n} y_n(t) = F(t, y_1, y_2, \ldots, y_n),$$

with initial conditions are:

$$y_i^k(0) = c_k^i, \quad 0 \leqslant k \leqslant m_i, \quad m_i < \alpha_i \leqslant m_{i+1}, \quad 1 \leqslant i \leqslant n.$$

According to the HPM, we construct following simple homotopies:

$$(1-q)D_*^{\alpha_i} y_i(t) + q\big[D_*^{\alpha_i} y_i(t) - y_{i+1}(t)\big] = 0, \quad i = 1, 2, \ldots, n-1,$$
$$(1-q)D_*^{\alpha_n} y_n(t) + q\big[D_*^{\alpha_n} y_n(t) - F(t, y_1, y_2, \ldots, y_n)\big] = 0,$$

or

$$D_*^{\alpha_i} y_i(t) + q\big[-y_{i+1}(t)\big] = 0, \quad i = 1, 2, \ldots, n-1, \tag{31}$$
$$D_*^{\alpha_n} y_n(t) + q\big[-F(t, y_1, y_2, \ldots, y_n)\big] = 0, \tag{32}$$

where $q \in [0, 1]$ is an embedding parameter. In case $q = 0$, Eqs. (31) and (32) become a linear equations, then we can easily solve. In case $q = 1$, Eqs. (31) and (32) turns out to be the original one, Eqs. (11) and (12).

In view of homotopy perturbation method, we use the homotopy parameter q to expand the solutions:

$$y_i(t) = y_{i,0} + qy_{i,1} + q^2 y_{i,2} + q^3 y_{i,3} + \cdots, \quad i = 1, 2, \ldots, n-1, \tag{33}$$
$$y_n(t) = y_{n,0} + qy_{n,1} + q^2 y_{n,2} + q^3 y_{n,3} + \cdots. \tag{34}$$

The approximate solution can be obtained by setting $q = 1$ in Eqs. (33) and (34):

$$y_i(t) = y_{i,0} + y_{i,1} + y_{i,2} + y_{i,3} + \cdots, \quad i = 1(1)n-1 \tag{35}$$

and

$$y_n(t) = y_{n,0} + y_{n,1} + y_{n,2} + y_{n,3} + \cdots. \tag{36}$$

Substituting from (33) and (34) into (31) and (32) respectively, then equating the terms with the identical powers of q, we can obtain a series of linear equations. These linear equations are easy to solve by using Mathematica software or by setting a computer code to get as many equations as we need in the calculation of the numerical as well as explicit solutions.

5.2. *Illustrative examples by HPM*

Example 1. Consider the same initial value problem in case of the inhomogeneous Bagley–Torvik equation [7] in Eq. (15a) which reduced to the following system of FDE (15b).

According to the HPM, we construct following simple homotopy:

$$D_*^{1.5} y_1(t) + q\big[-y_2(t)\big] = 0, \quad y_1(0) = y_1'(0) = 1, \tag{37}$$
$$D_*^{0.5} y_2(t) + q\big[y_2(t) + y_1(t) - 1 - t\big] = 0, \quad y_2(0) = 0. \tag{38}$$

Substituting from (33) and (34) into (37) and (38) respectively, and equating the terms with the identical powers of q, we can obtain the following series of linear equations.

$$q^0: \quad D_*^{1.5} y_{1,0}(t) = 0, \quad y_1(0) = y_1'(0) = 1, \tag{39.i}$$
$$D_*^{0.5} y_{2,0}(t) = 0, \quad y_2(0) = 0, \tag{39.ii}$$

$$q^1: \quad D_*^{1.5} y_{1,1}(t) = y_{2,0}(t), \tag{40.i}$$
$$D_*^{0.5} y_{2,1}(t) = -y_{2,0}(t) - y_{1,0}(t) + 1 + t, \tag{40.ii}$$

$$q^2: \quad D_*^{1.5} y_{1,2}(t) = y_{2,1}(t), \tag{41.i}$$
$$D_*^{0.5} y_{2,2}(t) = -y_{2,1}(t) - y_{1,1}(t). \tag{41.ii}$$

The solution of Eq. (39) using the initial conditions are:

$$y_{1,0} = 1 + t, \qquad y_{2,0} = 0.$$

After substituting by $y_{1,0}$ and $y_{2,0}$ in (39) we can find the solution of (39) in the form:

$$y_{1,1}(t) = J^{1.5} y_{2,0} = 0, \qquad y_{2,1}(t) = J^{0.5}(-y_{2,0} - y_{1,0}) + J^{0.5}(1 + t) = 0.$$

And so on, we can find that

$$y_{1,p}(t) = 0, \quad p \geqslant 1 \quad \text{and} \quad y_{2,p}(t) = 0, \quad p \geqslant 0.$$

In view of the above terms, we find $y_1(t) = 1 + t$, and $y_2(t) = 0$. So $y(t) = 1 + t$ is the required solution of (15).

Example 2. Consider the same initial value problem in Eq. (23a) which reduced to the following system of FDE (23b).

According to the HPM we construct following simple homotopy:

$$D_*^{2.5} y_1(t) + q\big[y_2(t)\big] = 0, \quad y_1(0) = y_1'(0) = 0, \ y_1''(0) = 2, \tag{42}$$
$$D_*^{0.5} y_2(t) + q\big[y_2(t) + y_1^2(t) - t^4\big] = 0, \quad y_2(0) = 0. \tag{43}$$

Substituting from (33) and (34) into (42) and (43) respectively, then equating the corresponding terms with the identical powers of q, we can obtain the following series of linear equations.

$$q^0: \quad D_*^{2.5} y_{1,0}(t) = 0, \quad y_1(0) = y_1'(0) = 0, \ y_1''(0) = 2, \tag{44.i}$$
$$D_*^{0.5} y_{2,0}(t) = 0, \quad y_2(0) = 0; \tag{44.ii}$$

$$q^1: \quad D_*^{2.5} y_{1,1}(t) = -y_{2,0}(t), \tag{45.i}$$
$$D_*^{0.5} y_{2,1}(t) = -y_{2,0}(t) - y_{1,0}^2(t) + t^4, \tag{45.ii}$$

$$q^2: \quad D_*^{2.5} y_{1,2}(t) = -y_{2,1}(t), \tag{46.i}$$
$$D_*^{0.5} y_{2,2}(t) = -y_{2,1}(t) - 2y_{1,0}y_{1,1}(t). \tag{46.ii}$$

The solution of Eqs. (44) using the initial conditions are:

$$y_{1,0} = t^2, \qquad y_{2,0} = 0.$$

After substituting by $y_{1,0}$ and $y_{2,0}$ in (45) we can find the solution of (45) in the form:

$$y_{1,1}(t) = J^{2.5}(-y_{2,0}) = 0, \qquad y_{2,1}(t) = J^{0.5}\left(-y_{2,0} - y_{1,0}^2\right) + J^{0.5}\left(t^4\right) = 0.$$

And so on, we can find that

$$y_{1,p}(t) = 0, \quad p \geqslant 1 \quad \text{and} \quad y_{2,p}(t) = 0, \quad p \geqslant 0.$$

In view of the above terms, we find $y_1(t) = t^2$, and $y_2(t) = 0$. So $y(t) = t^2$ is the required solution of (23).

In order to illustrate the advantages and the accuracy of the homotopy perturbation method for solving the systems (15b) and (23b), we have applied the method and using the first order perturbation, i.e. the approximate solutions are

$$y_1(t) = y_{1,0} + y_{1,1} \quad \text{and} \quad y_2(t) = y_{2,0} + y_{2,1}. \tag{47}$$

6. Conclusion

In this Letter, the VIM which is based on Lagrange multiplier method and the HPM are used to solve numerically multi-order fractional differential equation. We achieved a very good approximation with the actual solution of the equation by using one term of the iteration scheme derived above in both methods. It is evident that even using few terms of the iteration formula, the overall results getting very close to exact solution, errors can be made smaller by take new terms of the iteration formulas. A clear conclusion can be draw from the numerical results that the VIM and HPM are highly accurate numerical techniques without spatial discretization for nonlinear partial differential equations. They are powerful mathematical tools for solving wide classes of multi-order fractional differential equation. Finally, we point out that the corresponding analytical and numerical solutions are obtained using Mathematica 5.

References

[1] M.A. Abdou , A.A. Soliman, Physica D 211 (2005) 1.
[2] M.A. Abdou, A.A. Soliman, J. Comput. Appl. Math. (2004) 1.
[3] R.L. Bagley, R.A. Calico, J. Guidance Control Dynamics 14 (2) (1999).
[4] D.A. Benson, S.W. Wheatcraft, M.M. Meerschaert, Water Resour. Res. 36 (6) (2000) 1403.
[5] D.A. Benson, S.W. Wheatcraft, M.M. Meerschaert, Water Resour. Res. 36 (6) (2000) 1413.
[6] V. Daftardar-Gejji, H. Jafari, Appl. Math. Comput. 189 (1) (2007) 541.
[7] K. Diethelm, N.J. Ford, Appl. Math. Comput. 154 (2004) 621.
[8] M.M. Djrbashian, A.B. Nersesian, Izv. Acad. Nauk Armjanskoy SSR 3 (1) (1968) 3 (in Russian).
[9] J.H. He, Int. J. Mod. Phys. B 20 (10) (2006) 1141.
[10] J.H. He, Int. J. Non-Linear Mech. 34 (1999) 699.
[11] J.H. He, X.-H. Wu, Chaos Solitons Fractals 29 (2006) 108.
[12] F. Huang, F. Liu, ANZIAM J. 46 (2005) 317.
[13] M. Ichise, Y. Nagayanagi, T. Kojima, J. Electroanal. Chem. Interfacial Electrochem. 33 (1971) 253.
[14] A.A. Kilbas, M. Saigo, R.K. Saxena, J. Integral Equations Appl. 14 (4) (2002) 377.
[15] R.C. Koeller, Acta Mech. 58 (1984) 299.
[16] R.C. Koeller, J. Appl. Mech. 51 (1984) 299.
[17] F. Liu, V. Anh, I. Turner, P. Zhuang, J. Appl. Math. Comput. (2003) 233.
[18] B. Mandelbrot, IEEE Trans. Inform. Theory 13 (2) (1967) 289.
[19] R.J. Mark, M.W. Hall, IEEE Trans. Acoust. Speech Signal Process. 29 (1981) 872.
[20] K.S. Miller, B. Boss, An Introduction to the Fractional Calculus and Fractional Differential Equations, Wiley, New York, 1993.
[21] K.B. Oldham, J.C. Spanier, The Fractional Calculus, Academic Press, San Diego, CA, 1974.
[22] I. Podlubny, Fractional Differential Equations, Academic Press, San Diego, CA, 1999.
[23] S.B. Skaar, A.N. Michel, R.K. Miller, IEEE Trans. Automat. Control 3 (4) (1988) 348.
[24] N.H. Sweilam, M.M. Khader, Chaos Solitons Fractals 32 (2007) 145.
[25] N.H. Sweilam, J. Comput. Appl. Math. 207 (2007) 64.
[26] H.H. Sun, N. Onaral, Y. Tsao, IEEE Trans. Ciomed. Eng. 31 (10) (1984) 664.
[27] H.H. Sun, A.A. Abdelwahab, B. Onaral, IEEE Trans. Automat. Control 29 (5) (1984) 441.