Proposition 3.2.20 *Let I be any class and (X_i, τ_i) be a GT_4-space for all $i \in I$ and $f_i : X_i \to X$ be a surjective fuzzy open mapping for some $i \in I$. Then the final fuzzy topological space (X, τ) is also GT_4.*

Proof. Let F, G be disjoint closed subsets of X. Since f_i is surjective and continuous, then $f_i^{-1}(F)$, $f_i^{-1}(G)$ are also disjoint closed subsets of X_i. Because of that (X_i, τ_i) is normal it follows there are λ_i, $\mu_i \in L^{X_i}$ such that

$$\bigwedge_{z \in f_i^{-1}(F)} (\mathrm{int}_{\tau_i}\lambda_i)(z) \wedge \bigwedge_{w \in f_i^{-1}(G)} (\mathrm{int}_{\tau_i}\mu_i)(w) > \sup(\lambda_i \wedge \mu_i)$$

which means

$$\bigwedge_{x \in F} (\mathrm{int}_{\tau_i}\lambda_i)(f_i^{-1}(x)) \wedge \bigwedge_{y \in G} (\mathrm{int}_{\tau_i}\mu_i)(f_i^{-1}(y)) > \sup(\lambda_i \wedge \mu_i)$$

and this means

$$\bigwedge_{x \in F} (f_i(\mathrm{int}_{\tau_i}\lambda_i))(x) \wedge \bigwedge_{y \in G} (f_i(\mathrm{int}_{\tau_i}\mu_i))(y) > \sup(\lambda_i \wedge \mu_i).$$

Since f_i is fuzzy open, it follows $f_i(\mathrm{int}_{\tau_i}\lambda_i) \leq \mathrm{int}_{f_i(\tau_i)}(f_i(\lambda_i))$ for all $\lambda_i \in L^{X_i}$ and therefore

$$\bigwedge_{x \in F} (\mathrm{int}_{f_i(\tau_i)} f_i(\lambda_i))(x) \wedge \bigwedge_{y \in G} (\mathrm{int}_{f_i(\tau_i)} f_i(\mu_i))(y) > \sup(f_i(\lambda_i) \wedge f_i(\mu_i)).$$

Since $f_i(\lambda_i), f_i(\mu_i) \in L^X$, then we get that the final fuzzy topological space (X, τ) is normal. From Proposition 3.2.8 it follows that (X, τ) is GT_1-space and hence it is GT_4-space. □

The following result is a direct consequence of Propositions 3.2.19 and 3.2.20.

Corollary 3.2.10 The fuzzy topological quotient space and the fuzzy topological sum space of GT_4-spaces are also GT_4.

3.3 The Relation Between The GT_i-Spaces and The FT_i-Spaces

This section is devoted to show that our notion of GT_i-spaces is more general than the notion of FT_i-spaces, defined by Kandil and El–Shafee in [34], for $i = 0, 1, 2, 3, 4$.

Definition 3.3.1 [34] A fuzzy topological space (X, τ) is called:

(1) FT_0 if for all $x, y \in X$ with $x \neq y$ we have $x_\alpha \overline{q}\, \mathrm{cl}_\tau y_\beta$ or $\mathrm{cl}_\tau x_\alpha \overline{q}\, y_\beta$ for all $\alpha, \beta \in L_0$.

(2) FT_1 if for all $x, y \in X$ with $x \neq y$ we have $x_\alpha \overline{q}\, \mathrm{cl}_\tau y_\beta$ and $\mathrm{cl}_\tau x_\alpha \overline{q}\, y_\beta$ for all $\alpha, \beta \in L_0$.

(3) FT_2 if for all $x, y \in X$ and all $\alpha, \beta \in L_0$ we have $x_\alpha \overline{q}\, y_\beta$ implies there exist $\mathcal{O}_{x_\alpha}, \mathcal{O}_{y_\beta} \in \tau$ such that $\mathcal{O}_{x_\alpha} \overline{q}\, \mathcal{O}_{y_\beta}$.

(4) FT_3 if it is FT_1 and for all fuzzy points x_t and all closed fuzzy sets f with $x_t \overline{q}\, f$ there are $\mathcal{O}_{x_t}, \mathcal{O}_f \in \tau$ such that $\mathcal{O}_{x_t} \overline{q}\, \mathcal{O}_f$.

(5) FT_4 if it is FT_1 and for all $f, g \in \tau'$ with $f \overline{q} g$, there are $\mathcal{O}_f, \mathcal{O}_g \in \tau$ such that $\mathcal{O}_f \overline{q}\, \mathcal{O}_g$.

By *FT_i-space* we mean the fuzzy topological space which fulfills the axiom FT_i.

In the following proposition will be shown that the class of GT_0-spaces is larger than the class of FT_0-spaces.

Proposition 3.3.1 *Each FT_0-space is GT_0-space.*

Proof. Let (X, τ) be an FT_0-space and let $x, y \in X$ with $x \neq y$. Then from (1) in Proposition 1.2.1 it follows $x_\alpha \overline{q}\, y_\beta$ for all $\alpha, \beta \in L_0$ and thus $x_\alpha \overline{q}\, \mathrm{cl}_\tau y_\beta$. By (2) in

Proposition 1.2.1 we have $\mathcal{O}_{x_\alpha} \in \tau$ such that $\mathcal{O}_{x_\alpha} \bar{q}\, y_\beta$, that is, we have $f = \mathcal{O}_{x_\alpha} \in L^X$ with $y_\beta \leq f'$. Thus

$$f(y) \leq (1-\beta) \text{ and } \alpha \leq \text{int}_\tau f(x)$$

for all $\alpha, \beta \in L_0$. Taking $(1-\beta) < \alpha$ we get $f \in L^X$ and $\alpha \in L_0$ such that

$$f(y) < \alpha \leq \text{int}_\tau f(x).$$

Hence, (X, τ) is GT_0. □

The following example shows that there are GT_0-spaces which are not FT_0-spaces.

Example 3.3.1 Let $L = [0,1]$, $X = \{x, y\}$ with $x \neq y$ and let $\tau = \{\overline{0}, \overline{1}, x_{1/2}\}$. Then (X, τ) is a fuzzy topological space. Also, we have $x \neq y$ implies there is $f = x_1 \in L^X$ with $f(y) = 0 < 1/2 = \text{int}_\tau f(x)$, and thus (X, τ) is GT_0-space. Since the open fuzzy neighborhoods $\mathcal{O}_{x_1}$ of x_1 and the open fuzzy neighborhoods $\mathcal{O}_{y_1}$ of y_1 are only $\overline{1}$, it follows $x_1 \bar{q}\, y_1$ implies $\mathcal{O}_{x_1} q\, y_1$ and $\mathcal{O}_{y_1} q\, x_1$ for all $\mathcal{O}_{x_1}$ and $\mathcal{O}_{y_1}$. Hence, (X, τ) is not FT_0-space.

The following proposition shows that GT_1-spaces are more general than FT_1-spaces.

Proposition 3.3.2 *Each FT_1-space is GT_1-space.*

Proof. Similarly as in Proposition 3.3.1. □

Here, an example for GT_1-space which is not FT_1-space is given.

Example 3.3.2 Let $L = [0,1]$, $X = \{x, y\}$ with $x \neq y$ and let $\tau = \{\overline{0}, \overline{1}, x_{1/2}, y_{1/2}, x_{1/2} \vee y_{1/2}\}$. Then (X, τ) is a fuzzy topological space and there are $f = x_1 \in L^X$ and $g = y_1 \in L^X$ such that $x \neq y \in X$ implies

$$f(y) \;=\; 0 < 1/2 \;=\; \text{int}_\tau f(x)$$