From (1.2) and (1.7) we have

$$\lim\mathcal{F}(x) = \bigwedge_{A\in\rho_m(\mathcal{F})} \bigwedge_{f\in A} \bigvee_{\mathcal{M}\leq\mathcal{N}(x)} \mathcal{M}(f) \neq 1$$

or

$$\lim\mathcal{F}(y) = \bigwedge_{B\in\rho_m(\mathcal{F})} \bigwedge_{g\in B} \bigvee_{\mathcal{M}\leq\mathcal{N}(y)} \mathcal{M}(g) \neq 1.$$

But

$$\lim\mathcal{F}(x) = \bigwedge_{f\in A\in\rho_m(\mathcal{F})} \bigvee_{\mathcal{M}\leq\mathcal{N}(x)} \mathcal{M}(f) \neq 1$$

implies $\mathcal{M} \not\leq \mathcal{N}(x)$ and thus we have $\mathcal{M} \not\leq \mathcal{N}(x)$ or $\mathcal{M} \not\leq \mathcal{N}(y)$. Hence, (X,τ) is GT_2-space. □

The following example shows that there are GT_2-spaces which are not WT_2-spaces.

Example 3.4.3 Let $L = [0,1]$, $X = \{x,y\}$ with $x \neq y$ and let $\tau = \{\overline{0}, \overline{1}, x_1 \vee y_{1/4}, x_{1/4} \vee y_1, x_{1/4} \vee y_{1/4}\}$. Then (X,τ) is a fuzzy topological space and there are $f = x_1 \vee y_{1/4} \in L^X$ and $g = x_{1/4} \vee y_1 \in L^X$ such that

$$\mathrm{int}_\tau f(x) \wedge \mathrm{int}_\tau g(y) = 1 > 1/4 = \sup(f \wedge g),$$

that is, $\mathcal{N}(x) \wedge \mathcal{N}(y)$ does not exist and thus (X,τ) is GT_2-space. Since $\tau' = \{\overline{0}, \overline{1}, x_{3/4}, y_{3/4}, x_{3/4} \vee y_{3/4}\}$ and $\mathrm{cl}_\tau x_1 = \mathrm{cl}_\tau y_1 = \overline{1}$, then (X,τ) is not WT_1-space and since every WT_2-space is WT_1-space, then (X,τ) is also not WT_2-space.

3.5 The Relation Between The GNT_i-Spaces and The NT_i-Spaces

Now, we are going to study the relation between the GNT_i-spaces and the NT_i-spaces for $i = 0, 1, 2$ defined in [52].

Definition 3.5.1 [52] A fuzzy neighborhood space $(X, \mathcal{V})$, in sense of Lowen [48], is called:

(1) NT_0 if for all $x, y \in X$ with $x \neq y$ there are $f \in \mathcal{V}(x)$, $g \in \mathcal{V}(y)$ such that $f(y) \wedge g(x) < \varepsilon$ for all $\varepsilon \in L_0$.

(2) NT_1 if for all $x, y \in X$ with $x \neq y$ there is $f \in \mathcal{V}(x)$ such that $f(y) < \varepsilon$ for all $\varepsilon \in L_0$.

(3) NT_2 if $\operatorname{adh} \mathcal{V}(x) = x_1$ for all $x \in X$, where $\operatorname{adh} \mathcal{V}(x)$ is the fuzzy subset of X defined ([48]) by

$$\operatorname{adh}\mathcal{V}(x)(y) = \bigwedge_{f \in \mathcal{V}(x)} \Big(\bigwedge_{g \in \mathcal{V}(y)} \sup(f \wedge g) \Big) \tag{3.1}$$

for all $y \in X$.

By *NT_i-space* we mean the fuzzy neighborhood space which fulfills the axiom NT_i.

Recall here that the fuzzy neighborhood space $(X, \mathcal{V})$ in sense of Lowen can be identified with the third type of global fuzzy neighborhood spaces (X, h) which is called global prefilter-related fuzzy neighborhood spaces ([27]). This identification is given in Section 1.6 of this thesis by (1.22) and (1.23).

The following proposition shows that the fuzzy separation axiom GNT_0 for global fuzzy neighborhood spaces is more general than the fuzzy separation axiom NT_0 for fuzzy neighborhood spaces in sense of Lowen.

Proposition 3.5.1 *For each fuzzy neighborhood space $(X, \mathcal{V})$ in sense of Lowen which is NT_0, the global prefilter-related fuzzy neighborhood space (X, h) identified with $(X, \mathcal{V})$, by (1.22) and (1.23), is GNT_0.*

Proof. Let $(X, \mathcal{V})$ be an NT_0-space and let $x \neq y$ in X. Then there are $f \in \mathcal{V}(x)$, $g \in \mathcal{V}(y)$ such that $f(y) \wedge g(x) < \varepsilon$ for all $\varepsilon \in L_0$. From (1.23) we get

$$\dot{y}(f) = f(y) < \bigvee_{g \in \mathcal{V}(x), g \wedge \bar{\alpha} \leq f} \alpha = h(\dot{x})(f).$$

Hence, $\dot{y} \not\leq h(\dot{x})$ and therefore (X, h) is GNT_0-space. □

In the following we introduce an example showing that there is a GNT_0-space and the identified fuzzy neighborhood space in sense of Lowen is not NT_0.

Example 3.5.1 Let (X, τ) be the fuzzy topological space given in Example 3.3.1 and let (X, h) be the global fuzzy neighborhood space identified with (X, τ). By means of Example 3.3.1 we have (X, τ) is GT_0-space and thus by Proposition 2.2.1 (X, h) is GNT_0-space. Let $(X, \mathcal{V} = (\mathcal{V}(x))_{x \in X})$ be the fuzzy neighborhood space in sense of Lowen identified with (X, h). For all $f \in \mathcal{V}(x)$, $g \in \mathcal{V}(y)$ we have from (1.22) that

$$\text{int}_\tau f(x) \;=\; \text{int}_\tau g(y) \;=\; 1$$

and thus $f = g = \overline{1}$. Hence, $f(y) \wedge g(x) = 1$, that is, there is $\varepsilon = 1$ such that for all $f \in \mathcal{V}(x)$, $g \in \mathcal{V}(y)$ we have $f(y) \wedge g(x) = \varepsilon$ and therefore $(X, \mathcal{V})$ is not NT_0-space.

In the following proposition will be shown that the notion GNT_1 for global fuzzy neighborhood spaces is more general than the notion NT_1 for fuzzy neighborhood spaces in sense of Lowen.

Proposition 3.5.2 *Let $(X, \mathcal{V})$ be a fuzzy neighborhood space in sense of Lowen which is NT_1. then the global prefilter-related fuzzy neighborhood space (X, h) identified with $(X, \mathcal{V})$ is also GNT_1.*

Proof. Similarly as in Proposition 3.5.1. □

For the case of NT_1-spaces we have the following counterexample.

Example 3.5.2 Let (X, h) be the global fuzzy neighborhood space identified with the fuzzy topological space (X, τ) given by $X = \{x, y\}$ with $x \neq y$ and $\tau = \{\overline{0}, \overline{1}, x_1, y_1\}$. Then by Proposition 2.3.1, (X, h) is GNT_1-space.

But from (1.22) we have $\mathcal{V}(x) = \{(x_1 \vee y_\alpha) \in L^X \mid \alpha \in L\}$ and $\mathcal{V}(y) = \{(x_\beta \vee y_1) \in L^X \mid \beta \in L\}$ and thus $(x_1 \vee y_\alpha)(y) = \alpha$, $(x_\beta \vee y_1)(x) = \beta$ for all $\alpha, \beta \in L$. Taking