CHAPTER 1

INTRODUCTION

1- INTRODUCTION

One of the greatest remarks in the history of celestial mechanics was the pioneer research of the great French mathematician Louis Joseph Lagrange (1736-1823). His fame came in 1772 from his research [45] for the particular solutions of the Three-Body Problem, in his prize memoir "Essai d'une nouvelle method pour resoudre le probleme des trois corps". Lagrange solved the problem of the one body motion around a fixed center (two-body problem) taking into account the forces which produce the perturbation to the Newtonian law of gravitation. Nevertheless, there are more problems to be considered for the motion of the artificial Earth satellites. In the case of the Three Body-Problem, Lagrange was faced by a quintic (fifth degree) equation. The quintic equation appearing in the stationary solution was called Lagrange's equation. These problems in the artificial Earth satellites and the exact solution of the quintic equation associated with Lagrange's planetary equation are our aim of the present work.

1.1 The Earth Artificial Satellites

The mere anticipation of artificial satellites had already motivated several celestial mechanics theoreticians to turn their attention to the dynamical motion of these artificial objects. The whole succession of analyses was generated for an Earth satellite theoretical orbital motion, employing almost every known perturbation technique. The similarity of an artificial Earth satellite to our natural satellite, the Moon, implies that the wealth of Lunar theory, developed over many years, can be applied with proper interpretation to the motion of an artificial Earth satellite [54].

There are some essential differences between the Lunar motion and that of the artificial Earth satellites. The Moon is sufficiently remote from the Earth to be entirely free of frictional losses produced by motion through the Earth's atmosphere. Moreover, because of the Moon's distance, the orbital perturbations induced by the non-perfect Earth's shape are not as large, in a relative sense, as those induced on the motion of the artificial satellites.

Artificial Earth satellites travel in bound geocentric orbits, which are disturbed by the departure of Earth's shape from a sphere, atmospheric drag, the lunar and solar attractions, the solar radiation pressure, the Earth's magnetic field etc., i.e their orbits are not truly ideal Keplerian orbits. It is impossible to obtain a precise orbit without the knowledge of all the perturbations involved.

(I) Perturbation

Perturbations are classified as either secular or periodic. Secular perturbations are those which increase proportionally with time or to any power of the time. Secular terms in perturbation expressions are important, if they occur, because their presence may imply an unstable dynamical system [54]. Periodic terms are further divided into short-period and long-period perturbations. There are different kinds of perturbation theories depending upon the mode of calculating and expressing the perturbations.

If the perturbation effects are expressed in an analytical form, from which values can be computed at any time by assigning a value to (t), they are called *analytic perturbations*. The absolute and general terms are also used for this class. General perturbation techniques are extremely useful for theoretical investigations, because they lead to theories of orbital motion which determine the position of the body at each instant of time, or the time variation of the body's orbital elements. Various methods are used to obtain the general perturbation, e.g.:

(i) The method of the variation of parameters which exhibits the basic ideas and results of general perturbation theory.

(ii) Nonseparable Hamilton-Jacobi equation such that

$$H_1 = H - H_0 + O(k)$$

where k is a small parameter. Then the solution for H can be considered as a perturbed motion from the known solution of H_0, which is the Hamiltonian for the undisturbed motion.

(iii) The Von ZEIPEL method (Ralph [54]) which depends on obtaining a determining function with the desired properties for transforming canonical variables, so that the new Hamiltonian does not explicitly involve all the new angle variables:

In terms of the Delaunay variables, the Hamiltonian H for the disturbed dynamical system can be written in the form

$$H = H_0 + H_1 + \dots$$

with

$$H_0 = -\frac{\mu^2}{2L^2}, \qquad H_1 = \sum_{J,K} A_{JK}(L,G,H)\cos(Jl + Kg)$$

and (l, g, h, L, G, H) are the Delaunay variables defined as the following

l = mean anomaly, $L = \sqrt{\mu a}$,

g = argument of perigee, $G = L\sqrt{1-e^2}$,

h = Longitude of the node, $H = G \cos i$.

The analytical problem was to calculate the coefficient of each of the periodic terms. These calculations implied that the individual terms can be separated from the Hamiltonian H. Ralph [54] mentioned that Delaunay was able to accomplish this separation by using a succession of canonical transformations, each of which accomplished the removal of one periodic term from H. In the limit, the remaining H contains secular terms depending only on L,G and H. For his Lunar theory, as given in [54], Delaunay developed an expression for the disturbing function which contained 300 periodic terms removal of which required 500 operations involving canonical transformations.

Ralph [54] mentioned that in 1959 Brouwer illustrated a technique for combining a method of Von Zeipel with that of Delaunay and adapted the result to the theory of the motion of artificial Earth satellites. The periodic terms are divided into two classes: the short period terms which contain the mean anomaly in their arguments, and the long period terms whose arguments contain multiples of the mean argument of perigee. The principle of the method is to introduce a canonical transformation which eliminates one, or more variables in the Hamiltonian.

Computations using analytical perturbation equations are often lengthy and complex. Therefore, special perturbation techniques have been developed for celestial mechanics which have found widespread application in the calculation and determination of space vehicle orbits. The main disadvantage of the special perturbation techniques is that the positions of the orbiting body must be obtained by numerical integration rather than from an algebraic relation [54]. This means that the method of calculation of perturbations requires numerical integration for the differential equations of motion for the entire period of time which is being studied [18].

Special perturbations can be obtained by various methods. It is possible to resort to the integration of the differential equations of motion in rectangular coordinates and to determine the perturbed coordinates themselves. This was done for the first time by Cowell and Crommelin and alternately by Encke as given in Ref. [18]. It is possible to use polar coordinates and the fundamental method in which these are used is due to Honsen. Dubyago [18] mentioned that Honsen's method combines the variation of coordinates with variation of the elements.

It is well known that the actual shape of the Earth is that of an egg planet. The center of mass does not lie on the spin axis, and neither the meridian nor the latitudinal contours are circles. The net results of this irregular shape is to produce a variation in the gravitational acceleration relative to that predicted using a point mass distribution. Due to the spheroid of the central body, a component of force (transverse component) is produced. This force acts along the tangent to the instantaneous meridian, and always points towards the equator. The magnitude of this transverse component depends on the equatorial bulge. It reaches its maximum values at 45° latitude, and approaches zero at latitudes 0° and 90°. The motion can be visualized best by resolving it into individual motions along the meridian and latitudinal contours. The motion along a meridian can be imagined as consisting of a number of periodic motions, called zonal harmonics, of different frequencies and amplitudes. Similarly the motion along a latitudinal contour can be visualized as consisting, of a number of periodic motions called tesseral harmonics of different frequencies and amplitudes also. The zonal harmonics describe the deviations of a meridian from a great circle, whereas the tesseral harmonics describe the deviations of a latitudinal contour from a circle.

The Earth's gravitational potential is usually expressed by the following (Vinti's potential),

$$V = -\frac{\mu}{r}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\left(\frac{R}{r}\right)^{n}(C_{nm}\cos m\lambda + S_{nm}\sin m\lambda)\,P_{n}^{m}(\sin\delta) \tag{1.1}$$

where R is the equatorial radius of the Earth, μ is the Earth's gravitational constant, (r,λ,δ) are the geocentric coordinates of the satellite with λ measured east of Greenwich, C_{nm} and S_{nm} are harmonic coefficients, and $P_n^m(\text{Sin}\delta)$ are associated with Legender polynomials. In the potential function (1.1), the terms with $0 < m < n$ correspond respectively to zonal tesseral and sectorial harmonics. The case of axial symmetry is expressed by taking m = 0, but if equatorial symmetry is assumed, we consider only even harmonics, since $P_{2n+1}(\text{Sin}\delta) = - P_{2n+1}(-\text{Sin}\delta)$. Also, the coefficients C_{21} and S_{21} are vanishingly small. Further, if the origin is taken at the center of mass, the coefficients C_{10}, C_{11} and S_{11} will be equal zero. Therefore, in actual practice, Eq. (1.1) is better to be written as:

$$V = -\frac{\mu}{r} + \mu\sum_{n\geq 2} R^n J_n \frac{P_n(\sin\delta)}{r^{n+1}} -$$
$$- \mu\sum_{n\geq 2}\sum_{m=1}^{n} R^n (C_{nm}\cos m\lambda + S_{nm}\sin m\lambda)\frac{P_n^m(\sin\delta)}{r^{n+1}}$$

The Earth's oblateness is the major controller of an Earth satellite orbit. As shown in the last Eq., this can be expressed in the form

$$U = -\frac{\mu}{r}\sum_{n=2}^{\infty} J_n \left(\frac{R}{r}\right)^n P_n(\sin\beta)$$

where (as mentioned above) μ is the Gaussian constant multiplied by the mass of the Earth, R is the mean equatorial radius of the Earth, β is the terrestrial latitude, P_n is the Legendre polynomial of order n, and J_n is a constant coefficient [29].

Various investigators have treated the problem of the effect of oblations on the motion of artificial satellites in terms of canonical or non-canonical variables:

(i) Canonical variables

Sterne [68] presents an exact analytical solution by Hamilton-Jacobi's method of a Hamiltonian that contains the major part of the oblation effects in the planet's gravitational potential. The resulting unperturbed orbit is thus nearly correct, even for orbits of high inclination and eccentricity, than any elliptical motion.

Garfinkel [21] defined an approximate orbit by the potential function

$$V_0 = -\frac{\mu}{r} + 3K(\sin^2\theta - \cos^2 i)/a(1-e^2)r^2$$

with

$$\mu = 1 - 3K(1 - \frac{3}{2}\sin^2 i)/a^2(1-e^2)^{3/2}$$

where r is the radius vector, θ is the complement of the polar angle, (a,e,i) are constants analogous to elliptic elements, and K is a constant having the value 0.55×10^{-3} for the Earth. The above choice of V_0 leads to a simple closed solution, free from secular perturbations. Before Garfinkle the potential function V_0 was chosen by Brouwer (1946) in the form $V_0 = 1/r$ which leads to a Keplerian ellipse for the undisturbed orbit. But for Earth satellite such a choice of V_0 leads to large perturbations because of proximity of the equatorial bulge. In particular, the line of nodes may be regressed by 0.5° per revolution.

Garfinkle [22] gave the solution of the problem of motion in vacuum of an artificial satellite of the Earth under the assumption of the axial and the equatorial symmetry. The gravitational potential is given by

$$V = -\frac{1}{r} + \frac{2}{3} J P_2 (\sin\theta) / r^3 - \frac{8}{35} D P_4 (\sin\theta) / r^5$$

where r is the radius vector, θ is the declination, P_2, P_4 are the Legendre polynomials and (J,D) are quantities of first and second order respectively.

Kozai [42] derived perturbation of the six orbital elements of a close Earth satellite moving in the gravitational field of the Earth as function of mean orbital elements and time. He also proved that there are no long-periodic terms of the first order in the expression of the semi-major axis.

Brouwer [5] gave the solution of the main problem for a spheroidal Earth with the potential limited to the principal term and the second harmonic which contains the small factor (K_2). The solution is developed in powers of K_2 in canonical variables. The periodic terms both of short and long period are developed to first order, the secular motions are obtained to second order. The results are obtained in a closed form. The solution does not apply to orbits near to the critical inclination 63.4°, but it is, otherwise valid for any inclination. Also, he studied the perturbing effects of the third through the fifth zonal harmonics and gave the long period terms and the additions to the secular motion caused by the further harmonic in the potential.

Garfinkel and Gregory [23] extended the known solution of the main problem of the artificial satellite to include the effects of all the higher zonal harmonics of the geopotential.

This problem in its Hamiltonian form was treated by Claes [9]. His method is based on eliminating angular variables from the Hamiltonian function. Lie transformations are used, to remove short periodic terms, then long periodic terms. The general solution up to J_2^3 is represented by the generators of the transformations.

Since the early studies of Vinti (1959-1961) and Aksenov et al (1963) an efficient method of the motion analysis of artificial satellite is the two fixed-centers method. This is for three major reasons:

1- The motion of a satellite in the gravitational field of two fixed point masses is integrable.
2- The harmonics of the Earth's potential are, by far, the largest source of perturbation of the motion of artificial satellites.
3- The relative difference between the Earth's potential and that of a single center is of order of 10^{-3}, but two suitable fixed centers give a much better approximation of the order of 10^{-6}.

Under these conditions the solutions of the two-fixed centers problem can be used as an excellent first approximation of the motion of artificial satellites. From this study Marchal [48] shows that: with the Hamiltonian parameters developed for the two -fixed centers problem, a simple and very accurate expression of the 'quasi-integral' can be given for the motion of artificial satellites perturbed by the Earth's zonal harmonics. This motion can be considered as integrable. A theoretical analysis showed that the relative difference between the true motion and the corresponding integrable motion remains forever less than 10^{-14}, for all regular orbits, even in the vicinity of critical inclinations.

Zafiropoulos [78] integrated the equation of the variation of the oscillating elements of a satellite moving in an axi-symmetric gravitational field to yield the complete first order perturbation for the elements of the orbit. The expressions obtained include the effects produced by the second to eighth spherical harmonics.

The perturbation in the position of the satellite due to the Earth's gravitational effects in the radial, transverse and normal components was given by Rosborough and Tapley [55].

Economical and stable recurrence formula for the Earth's zonal potential and its gradient KS regularized theory have been established by Sharaf and Awad [59] for any number of zonal harmonic coefficient.

The drawbacks of the canonical formulation were in the areas of:

(1) The extensive and confusing notation required in the solution.

(2) The art of determining and selecting the generating function that has a desired form and result.

(ii) Non-canonical variables.

The problem of the motion of a satellite, subject to a force resulting from an inverse-square gravitational attraction and a perturbation due to the Earth's oblations and constrained to Lie in equatorial plane was treated in 1983 by Jezewski [40]. He gave an analytic solution for the J_2 perturbed equatorial orbit in terms of elliptic and integral functions, by using a non-canonical approach to obtain a uniform analytical solution for the problem. The basis for the solution was the transformation and uncoupling of the differential equations that describe the model. The resulting equations are amenable to analysis. There are a two fold utility for such a solution: The first leads to a clearer understanding of the motion by examining the simplest model. The second is to use the solution as an intermediary orbit for the motion subject to perturbations due to zonal harmonics and small inclinations.

Heimberger et al [31] treated the motion of artificial satellite in the gravitational field of an oblate body by using two Lie transformations to derive explicit result for the long periodic and secular perturbation for satellite's orbits.

In the previous methods, authors gave the perturbation of the zonal harmonic on the orbital elements, but Roth [56] gave a formula for the perturbation of the anomalistic period (the time interval between two successive pericentre passes of a satellite) of a highly eccentric orbit due to the zonal harmonics. This perturbation depends essentially only on the semi-major axis a, the eccentricity e, and the latitude δ of pericentre.

In the present work an improved analytical solution was obtained for the motion of an artificial satellite subject to an inverse-square gravitational force of attraction and a perturbation due to zonal harmonics through J_6. Our solution based on the perturbation techniques proposed by CUI DOU-XING [10]. He derived the perturbation of artificial satellite by a new method. The novel feature of the method was that the right hand term $f(t;x)$ of the differential equation needs not to be developed in series as it was performed by the Von Zeiple's method and other methods. In CUI solution, the disturbing function for the artificial satellite motion was limited to zonal harmonics J_4, while in the present work the disturbing function will be limited to zonal harmonics J_6 to obtain more accuracy to the value of the orbital elements $(a, e, i, \Omega, \omega, M_0)$.

1.2 The Three Body Problem

Although three body systems seem to be only a single step removed from the two-body problem, the three-body problem is of a much higher degree of complexity than the two-body problem. If rapid development of mathematical generalizations in which the basic qualitative nature of the operations has given more emphasis than the operations themselves. Thus, one can still derive qualitative properties of the solution characteristics from the nature of the equations themselves instead of solving differential equations. Poincare introduced this topological approach into studies of the fundamental behavior of the three-body motion. Although the topological viewpoint does not lead to quantitative solutions, it does provide a new outlook which generates a considerable deep insight into the fundamental nature of the general motions.

Solutions to the three-body problem are now obtained almost by numerical integration through the use of automatic digital computers. In principle, the general three-body motion could be investigated by numerical solution of a variety of configurations. This is not a practical technique because of the large number of cases that would require solution, coupled with the large cost of calculations, even using modern high-speed computer.

(I) The Restricted Problem

The restricted problem of three bodies occupies a central place in analytical dynamics, celestial mechanics, and space dynamics. Entry into celestial mechanics and space dynamic can be gained by the study of the problem of two bodies. This problem begins with Euler and Lagrange in (1772), continues with Jacobi (1836) and Hill (1878), and is followed by Poincare

(1899), Levi-Civita (1905), and Birkhoff (1915). The span of almost 200 years, from Euler until now, includes other great names and important contributions.

The difference between the general three body problem and the restricted problem are :

(i) The energy is conserved in the former and not conserved in the latter.

(ii) The general problem allows any sets of initial conditions for the three particles involved; the restricted problem requires circular orbits for the primaries.

(A) The circular restricted three body problem

In this problem, two bodies revolve around their center of mass in circular orbits under the influence of their mutual gravitational attraction and a third body (attracted by the previous two but not influencing their motion) moves in the plane defined by the two revolving bodies. The restricted problem of the three bodies is to describe the motion of this third body. If the problem is further restricted, the test particle being constrained to move in the orbital plane of the two massive bodies. This particular variation is called the coplanar restricted three-body problem.

(B) Modifications of The Restricted Problem

The restricted problem of the three bodies is a specified circular motion for m_1 and m_2 and the motion of m_3 must take place in the plane defined by the motion of m_1 and m_2. Here some modifications of this basic problem which are mentioned in [73].

(i) The motion of m_1 and m_2 is not circular. In this case m_1 and m_2 are moving in a conic section. The most important case is when m_1 and m_2 are moving in elliptical orbits which define then "elliptic restricted problem"

(ii) When the initial conditions are such that the third body is not initially in the plane of motion of m_1 and m_2, or when its initial vector velocity has a component not in the plane in which the three-dimensional restricted problem is concerned.

(iii) The value of the mass ratio m_1/m_2, denoted by (μ) have important effects on the motion of the third body and on the approach to the problem. When μ=0, the restricted problem changed to two-body problem, so problems with small values of (μ) appear as perturbation problems of the two-body problem. Specification of μ is important, the case μ=1/2 is essentially a three-body problem and is known as the Copenhagen problem. Poincare's restricted problem assumes a small value of μ so that perturbation theories become applicable.

(iv) If masses of the participating primaries vary with time, the basic equations of motion are to be significantly modified but the basic idea of the restricted problem is still useful in stellar dynamics.

(v) If the noncentral forces are involved in the problem, the important question is whether a circular or conic section motion of the primaries is still meaningful or not.

If so, the modified field becomes significant only when the motion of the third body is studied. This case has been studied by many authors, for example Sarris [58], Hadjidemetriou [28], (Kwok and Nacozy) [44], Timoshkova [75] and(Shrivastava and Ishwar) [61].

(II) **Lagrange's Equation**

It is well known that there are five equilibrium solutions in the restricted three-body problem. Three are collinear with the primaries and the other two are in equilateral triangle configuration with the primaries. Douboshin [16] began the systematic study of the n-rigid bodies problem proving the existence of ten integrals of motion. Afterwards a number of papers have been devoted to this subject, and the case of the three rigid bodies has been extensively studied.

In particular, the restricted problem of three rigid bodies m_1, m_2, m_3 is obtained as the study of the motion of infinitesimal mass m_3, under the action of the primaries m_1, m_2. This study was begun by Nikoloev [53], who obtained equilibrium solutions when the more massive primary m_1 is an oblate spheroid, and the other two are spheres. This work is extended by Sharma and Subbarao [60] in the case where m_1, m_2 are oblate spheroids, and where the infinitesimal m_3 is a sphere. Also, Douboshin [17] has given conditions for the existence of collinear and equilateral equilibrium solutions in the case of rigid bodies having an axis and a plane of symmetry. Other authors such as Elipe and Ferrer [19] continued the work of Duboshin giving explicit equations that allow to obtain the collinear and triangular solutions which in general are not equilateral.

In the planer case of the problem, these families have been computed both for collinear points (Henon, 1965) [32] and the triangular points, Goodrich [26] and Deprit [15] and their termination has been determined.

The families of the three dimension periodic solutions about the collinear equilibrium points have been studied for large values of the mass parameter (μ =0.4) by Bray and Goudas [3] and small (μ = 0.00095) by Zagouras and Kazantzis [79].

The three dimensional periodic oscillations about the triangular equilibrium points have been studied by series expansions (Buck [7], Heppenheimer [33], Erdi [20]) which are valid for small values of the orbital parameters used in each contribution i.e, they are valid in the vicinity of the equilibrium points.

The third order parametric expansions given by Buck in (1920) for the three dimensional periodic solutions, about the triangular equilibrium points of the restricted problem, are improved by fourth order terms. The corresponding family of periodic orbits have been computed for (μ= 0.00095) Zagouras [80] and their terminal has been determined.

Also, Gomez and Noguera [24] gave some numerical results about natural families of periodic orbits, which emanate from limiting orbits around the equilateral equilibrium points of the restricted three-body problem, when the mass ratio is greater than Routh's critical one. Delibaltas [14] shows that in the general planer three-body problem, for constant values of the three masses (m_1, m_2, m_3) a symmetric periodic orbit with a binary collision in a rotating frame of reference, is determined by the values of the abscissa x_{10} of m_1 and the energy E of the system. In the case of equal masses of the two bodies and small mass of a third body, there are several symmetric periodic collision orbits similar to the corresponding orbits in the restricted three-body problem. Many authors have studied the problem of periodic orbit. Hadjidemetriou [28] presented some families of periodic orbit when the masses of two of the bodies are small compared with the mass of the third body. Also, Markellos [49] outlines the procedure involved in the determination of any type of three dimensional periodic orbit and apply them in computing families of such orbits of the general three-body problem.

The three dimensional periodic motion about the positions of equilibrium of the Hill problem has been studied by Zagouras and Markellos [81]. Periodic solution are approximated via a fourth order parametric expansion with respect to an orbital parameter.

Kammeyer [41] established many of the symmetric periodic orbits in the rectilinear problem of three bodies with the middle mass much larger than the outer masses. He shows that for all of the families of orbits which have been explored numerically, stability holds for small ratios of the outer masses to the inner mass.

A large number of authors have already described the essential features of the planer elliptic restricted three-body problem, in order, mainly, to compare and contrast this problem with the better known circular three-body problem, e.g Broucke [4]. Also, Kwok and Nacozy [43] provide results concerning relations between families of periodic orbits in the circular and elliptic restricted problem. A systematic approach is taken to generate periodic orbits of the elliptic restricted problem from families of periodic orbits of the circular restricted problem.

This problem was studied by Gomez and Merce [25] when the mass ratio is equal to zero. When two bodies moving in orbits around the same primary body have mean motions whose ratio is close to that of two small integers.

When Lagrange planetary equations are integrated to give the solution in the form of a power series in the mass of the disturbing body, certain terms in the disturbing function, (the critical terms) give rise to terms of long period and large amplitude in the solution. For very close commensurable cases the amplitude of these perturbations can be so large that the higher order terms in the power series becomes important, and so some other methods of solving Lagrange equations are necessary. There is no known methods of solving the equations analytically in a closed form for a general case close to a commensurability, but certain particular solutions of the problem are known. There are 'periodic solutions' in which the three bodies periodically return to the same configuration in an appropriate uniformly rotating frame of reference. Many such solutions are known for the plane, and one of the secondary bodies has zero mass, and the other, the disturbing body, moves in a fixed circular orbit around the primary. Sinclair [63] states that various families of periodic solutions are shown to exist in the three-body problem in which the two secondary bodies are close to a commensurability in mean motions. Both the restricted problem and the planer non-restricted problem are considered.

Lyapunov [47] has shown that the Lagrangian solutions of the unrestricted three-body problem are unstable if the condition for stability is that the perturbed and unperturbed triangles formed by the attracting masses differ infinitesimally from each other, and that the sides of the triangle in the perturbed motion differ infinitesimally from the lengths which they would have had at the same moment of time in the unperturbed motion.

It is of interest to investigate the stability of triangular Lagrangian solutions of restricted problems of celestial mechanics. In fact, the triangular Lagrangian solutions of the restricted circular three-body problem possess stability in the original sense, namely, these solutions are stable in the first approximation if the condition for stability is that the coordinates of the vertices of the triangle in the perturbed motion differ infinitesimally from the coordinates of the unperturbed triangle at the same moment of time.

Stability exists if the smallest of the attracting masses satisfies the inequality $\mu < \mu_0 \approx 1/26$. Leontovich [46] using Arnol'd's results [1] has shown that the triangular particular solutions of the flat restricted circular three-body problem are stable in any approximation for almost all masses $\mu < \mu_0$.

Grebenikov [27] considered the stability in the Lyapunov sense of the Lagrangian triangle solutions in the restricted elliptic three-body problem. It was proved that for sufficiently small values of the orbit eccentricity of the attracting masses and some conditions imposed on the masses, the Lagrange solutions are stable if the condition for stability is that the perturbed and unperturbed triangles formed by the masses differ infinitesimally from each other.

Szebehely [73] showed that in the restricted problem the equilateral triangular points L_4, L_5 are stable for the mass ratio M of the finite bodies less than $\mu_0 = 0.03852$. Winter [76] proved that the stability of the equilateral points is due to the presence of the Coriolis force in the equation of motion. The effect of perturbation of the Coriolis force on the stability of the equilibrium points, keeping the centrifugal force constant has been examined by Szebehely [74]. The same problem was studied by Subbarao and Sharma [71], with one of the primaries as an oblate spheroid and its equatorial plane coinciding with the plane of motion. Bhatangar and Hallan [2] studied the effect of small perturbations in the Coriolis and the centrifugal forces on the location and stability of equilibrium points in the restricted problem. They proved that for the triangular points L_4, L_5, the range of stability increases or decreases depending upon whether the point $(\varepsilon, \varepsilon')$ lies in one or the other of the two parts in which the $(\varepsilon, \varepsilon')$ plane is divided by the line $36\varepsilon - 19\varepsilon' = 0$.

Ishwar and Singh [38] studied the effect of perturbations on the location of the equilibrium points in the restricted problem of three bodies with variable mass. Singh and Ishwar [64] examined the effect of perturbations in Coriolis and centrifugal forces on the stability of triangular points, in the restricted problem of three-bodies with variable mass.

Lagrange in a stationary solution of the three-body problem was faced by unsolvable quintic equation. At 1985, Ioakimidis [63] gave an analytical solution of this equation and verified his solution numerically. He applied an elementary real integral formula for the closed form solution of nonlinear equations. Two different approaches to this solution are obtained and numerical results (obtained by both approaches) are displayed when the distance between two masses m_1, m_2 was equal to one Astronomical unit and distance between m_2, m_3 was determined when the three masses m_1, m_2, m_3 take the values 1, 3, 5 respectively.

In this thesis we will follow the same procedure with different distance between m_1, m_2 i.e the distances between the masses m_1, m_2, and m_2, m_3 will be determined when the three masses take different values. We studied these two approaches carefully and used one to obtain the numerical results.

This thesis is divided into four main chapters following the literature survey which is given in chapter (1). The second chapter presents the basic concepts and definitions of the planetary motion.

The third and the fourth chapters deal with the problem of analytical theory of Earth's artificial satellite under the effect of the zonal perturbations. We treated it by using CUI DOU-XING technique to obtain the first and second order perturbations in the orbital elements. The expressions obtained include the effects produced by the second through six zonal harmonics. The secular terms of the first and second order and the periodic terms of the first order are obtained. Numerical examples in which the values of the orbital elements, $(a, e, i, \Omega, \omega, M_0)$ are found. In the first example of the Balloon satellite 1963, 30D, numerical values for the orbital elements at the beginning, middle of the lifetime of the Balloon and near the decay are computed. Another example of the Pageos 1 Balloon satellite was also considered where numerical values for the orbital elements were found, at the beginning, middle of the first two years of the lifetime and near the end of the second year. A comparison is then made between our results and the results obtained by Slowey and Susanna.

The fifth chapter describes the formulation and solution of the quintic equation appearing in a stationary solution of the three-body problem which is known as Lagrange equation. This equation was solved when the distance between two masses m_1, m_2 is not equal to one Astronomical unit, by using Ioakimidis method

For the first time a verification of this solution in the solar system was given with numerical examples.