if for each $\nu = \bar{\nu}$ the maximum allowable levels $(\mu_1, \mu_2, \ldots, \mu_{s-1}, \mu_{s+1}, \ldots, \mu_m)$ for the m-1 objectives $(f_1, f_2, \ldots, f_{s-1}, f_{s+1}, \ldots, f_m)$ are determined in a specified region, an efficient solution of $(LVMP)_{\bar{\nu}}$ can be found by solving $(P)_{\mu,\bar{\nu}}$. A systematic variation of μ_i's will yield a set of efficient solutions.

a) The Set of Feasible Parameters

The set of feasible parameters of problem $(LVMP)_\nu$

which is denoted by A is defined by

$$A = \{(\mu,\nu)\varepsilon R^{m+r-1} | M_s(\mu,\nu) \neq \phi\} \quad (3.1)$$

It follows from [22], that the set A is nonempty, unbounded, convex and if $M_s(\mu,\nu)$ is bounded for one $(\mu,\nu)\varepsilon A$, then A is closed.

It is clear that the set A is of more interest than the set

$$D = \{\nu\varepsilon R^r | M(\nu) \neq \phi\}, \quad (3.2)$$

which has the same properties as the set A, (see [22]). From the definitions of the sets A and D, it follows directly that

$$D = \{\nu\varepsilon R^r | (\mu,\nu)\varepsilon A\}$$

Use of simplex method.

b) The Solvability Set

The solvability set of problem $(LVMP)_\nu$, which is denoted by B, is defined by

$$B = \{(\mu,\nu)\varepsilon A \mid \text{problem } (LVMP)_\nu \text{ has efficient solutions}\} \tag{3.3}$$

It follows from [22], that if for one $(\mu,\nu)\varepsilon B$ it holds that the set $m_{opt}(\mu,\nu)$ is bounded, then $B = A$, where

$$m_{opt}(\mu,\nu) = \{x^* \varepsilon R^n \mid f_s(x^*) = \min_{x\varepsilon M_s(\mu,\nu)} f_s(x)\}, \tag{3.4}$$

and therefore under these conditions the set B is unbounded and convex [22].

A direct corollary of this result was given in [22], which states that, if $B \neq \phi$ and the set $M_s(\mu,\nu)$ is bounded for one $(\mu,\nu)\varepsilon A$, then $B = A$.

3.2 The Stability Set of the First Kind

Suppose that $(\bar{\mu},\bar{\nu})\varepsilon B$ with a corresponding efficient solution $\bar{x}$, then the stability set of the first kind of problem $(LVMP)_\nu$ corresponding to $\bar{x}$ denoted by $s(\bar{x})$ is defined by

$$s(\bar{x}) = \{(\mu,\nu)\varepsilon B \mid \bar{x} \text{ is an efficient solution of } (LVMP)_\nu\} \tag{3.5}$$

From [22], it follows that the set $s(\bar{x})$ is closed and star shaped [22], with a common point of visibility (μ^*,ν^*), where $\mu_i^* = f_i(\bar{x})$, $i = 1,2,\ldots,m$, $i \neq s$ and

$\nu_k^* = g_k(\bar{x})$, $k = 1,2,\ldots,r$. Moreover, we can have the following results.

Lemma 3.1

If the set N is defined as

$$N = \{(\mu,\nu)\varepsilon B \mid \begin{matrix} \mu_i \geq f_i(\bar{x}), & i = 1,2,\ldots,m, \quad i \neq s \\ \nu_k \geq g_k(\bar{x}), & k = 1,2,\ldots,r \end{matrix} \} \qquad (3.6)$$

then,

$$N = \{(\mu,\nu)\varepsilon B \mid \min_{x\varepsilon M_s(\mu,\nu)} f_s(x) \leq f_s(\bar{x})\}$$

The proof follows directly from the fact that $\bar{x}\varepsilon M_s(\mu,\nu)$ for every $(\mu,\nu)\varepsilon N$.

Corollary 3.1

From Lemma 3.1, it follows directly that $S(\bar{x}) \subseteq N$.

Let $S_1(\bar{x}) = \{\mu\varepsilon R^{m-1} \mid (\mu,\nu)\varepsilon S(\bar{x})\}$,

$S_2(\bar{x}) = \{\nu\varepsilon R^{r} \mid (\mu,\nu)\varepsilon S(\bar{x})\}$,

and let $E(\nu)$ denote the set of all efficient solutions of problem $(\text{LVMP})_\nu$.

Lemma 3.2

If $\bar{\nu} \varepsilon \bigcap_{i\varepsilon I} S_2(x^i)$, then $x^i\varepsilon E(\bar{\nu})$ for all $i\varepsilon I$, where I is an index set. The proof follows directly from the definitions.

Corollary 3.2

From Lemma 3.2, it follows that for all $\nu \varepsilon \bigcap_{i\varepsilon I} S_2(x^i)$, the points x^i, $i\varepsilon I$ are efficient solutions of the problem $(LVMP)_\nu$.

For all the above properties of the stability set of the first kind see [23].

a) Determination of the Stability Set of the First Kind

Let $(\bar{\mu},\bar{\nu})\varepsilon B$ with an efficient solution $\bar{x}$ and let

$$f_i(x) = \sum_{j=1}^{n} c_{ij}x_j, \quad i = 1,2, \ldots, m,$$

$$g_k(x) = \sum_{j=1}^{n} a_{kj}x_j, \quad k = 1,2,\ldots, r,$$

The corresponding Kuhn-Tucker conditions [19] will then have the form

$$\begin{aligned}
& c_{s\alpha} + \sum_{\substack{i=1\\ i\neq s}}^{m} \lambda_i c_{i\alpha} + \sum_{k=1}^{r} \delta_k a_{k\alpha} = 0, \quad \alpha = 1,2,\ldots,n \\
& f_i(\bar{x}) \leq \mu_i, \quad i = 1,2,\ldots,m, \quad i \neq s, \\
& g_k(\bar{x}) \leq \nu_k, \quad k = 1,2,\ldots,r, \\
& \lambda_i (f_i(\bar{x}) - \mu_i) = 0, \quad i = 1,2,\ldots,m, \quad i \neq s, \\
& \delta_k(g_k(\bar{x}) - \nu_k) = 0, \quad k = 1,2,\ldots,r, \\
& \lambda_i \geq 0, \quad i = 1,2,\ldots,m, \quad i \neq s \\
& \delta_k \geq 0, \quad k = 1,2,\ldots,r.
\end{aligned} \tag{3.7}$$

The first and the last two relations of (3.7) represent a polytope T for which its points can be determined using any algorithm which is based on the simplex method, for example Balinski [1]. According to whether any of the variables λ_i, $i = 1,2,\ldots,m$, $i \neq s$, δ_k, $k = 1,2,\ldots,r$ is zero or positive, the stability set of the first kind $s(\bar{x})$ will be determined.

Let $(\lambda^*, \delta^*) \varepsilon T$, where

$$\lambda_i^* = 0, \quad i \varepsilon I \subseteq \{1,2,\ldots,m\} - \{s\}$$

$$\delta_k^* = 0, \quad k \varepsilon J \subseteq \{1,2,\ldots,r\},$$

Then, the corresponding set of μs and νs which solves (3.7) is given as

$$s_{I,J}(\bar{x}) = \{(\mu,\nu) \varepsilon R^{m+r-1} \mid \begin{array}{l} \mu_i \geq f_i(\bar{x}),\ i \varepsilon I \\ \mu_i = f_i(\bar{x}),\ i \notin I \\ \nu_k \geq g_k(\bar{x}),\ k \varepsilon J \\ \nu_k = g_k(\bar{x}),\ k \notin J \,\} \end{array}$$

where either one of the index subsets I, J or both can be empty.

If L denotes the set of all possible ordered pairs (I,J) which result from the points of T as described before, then the set $S(\bar{x})$ is given as

$$S(\bar{x}) = \bigcup_{(I,J) \varepsilon L} s_{I,J}(\bar{x}) \tag{3.8}$$

The following points are useful in investigating the Kuhn-Tucker conditions (3.7) and the stability set of the first kind $S(\bar{x})$ given by (3.8).

(i) If it is found that the only possible index subset I corresponding to the points of T is $I = \phi$, then $\bar{x}$ is efficient solution of $(LVMP)_{\bar{\nu}}$.

(ii) If it is found that the only possible index subsets I and J corresponding to the points of T are $I = \phi$ and $J = \phi$, then

$$S(\bar{x})=\{(\hat{\mu},\hat{\nu})\}$$

which is a one point set (convex and closed), where $\hat{\mu}_i = f_i(\bar{x})$, $i = 1,2,\dots,m$, $i \neq s$ and $\hat{\nu}_k = g_k(\bar{x})$, $k = 1,2,\dots,r$.
In this case $\bar{x}$ will be efficient solution for the unconstrained vector minimization problem

$$\min[F(x),G(x)]$$

(iii) If as in case (i), $I = \phi$ and moreover, for all the points of T, it is found that $\delta_k > 0$, $k = 1,2,\dots,d$, $d < r$, then $\bar{x}$ will be efficient solution for the following vector minimization problem

$$\min[F(x),g_1(x),g_2(x),\dots,g_d(x)]$$

subject to

$$g_i(x) \leq \nu_i, \quad i = d+1, \dots, r$$

for all ν_i such that

$\nu_i \geq g_i(\bar{x})$, $i \varepsilon c = \{i \varepsilon \{d+1, \ldots, r\} \mid \delta_i = 0\}$

$\nu_i = g_i(\bar{x})$, $i \varepsilon \{d+1, \ldots, r\} - c$

3.3 The Stability Set of the Second Kind

Suppose that $(\bar{\mu},\bar{\nu}) \varepsilon B$ with a corresponding efficient solution $\bar{x}$ and $\bar{x} \varepsilon \sigma(\bar{\mu},\bar{\nu},I,J)$, where

$$\sigma(\bar{\mu},\bar{\nu},I,J) = \{x \varepsilon R^n \mid f_i(x) = \bar{\mu}_i, \quad i \varepsilon I \subset \{1,2,\ldots,m\} - \{s\}$$
$$f_i(x) < \bar{\mu}_i, \quad i \notin I, \tag{3.9}$$
$$g_k(x) = \bar{\nu}_k, \quad k \varepsilon J \subset \{1,2,\ldots,r\},$$
$$g_k(x) < \bar{\nu}_k, \quad k \notin J\}$$

Then, the stability set of the second kind of problem $(LVMP)_\nu$ corresponding to $\sigma(\bar{\mu},\bar{\nu},I,J)$ denoted by $Q(\bar{\mu},\bar{\nu},I,J)$ is defined by

$$Q(\bar{\mu},\bar{\nu},I,J) = \{(\mu,\nu) \varepsilon B \mid E'(\mu,\nu) \cap \sigma(\mu,\nu,I,J) \neq \phi\} \tag{3.10}$$

where $E'(\mu,\nu)$ is the set of efficient solutions of problem $(LVMP)_\nu$ corresponding to $(\mu,\nu) \varepsilon B$.

It is clear that $E'(\mu,\nu) \subset E(\nu)$.

a) An Algorithm for Decomposing the Solvability Set According to the Stability Sets of the Second Kind Together with the Determination of the Corresponding Efficient Points

Let us consider the following PMOLPP: