§ 3.4. Some Known Results:

This section is concerned with some known results about the relation between the Dirichlet, conditional Dirichlet and weak Dirichlet and the strong limit-point property of the second-order linear differential expression M(.).

In 1973 [11] Everitt, Giertz and Weidman proved that in the case $P \equiv 1$ on $[0, \infty)$ $M(\cdot)$ is on strong limit-point and satisfies the Dirichlet condition if $q \in L^r(0,\infty)$ for some $1 \leq r < \infty$.

In 1976 [14] Everitt took in his consideration the second-order linear differential expression M (·) under the conditions of Theorem 3.3.1 and he proved the following result :

Theorem 3.4.1 **(Everitt, [14])**

Let all conditions of theorem 3.3.1 hold on the interval $[0, \infty)$. Then

(i) If M (.) is strong limit-point at ∞ then

$$\lim_{x \to \infty} \int_0^x p|f'|^2 + q|f|^2 \text{ exists and is finite}$$

for all $f \in \Delta$

if $p^{-1} \notin L(0,\infty)$ and the above limit condition is satisfied, then M(.) is strong limit-point at ∞.

(ii) $M(\cdot)$ is conditional Dirichlet at ∞ if and only if

$$\lim_{x \to \infty} \int_0^x q|f|^2$$ exists and is finite for all $f \in \Delta$

(iii) $M(\cdot)$ is Dirichlet at ∞ if and only if

$$\lim_{x \to \infty} \int_0^x |q|\,|f|^2 < \infty ,$$

i.e. $|q|^{\frac{1}{2}} f \in L^2(0,\infty)$ for all $f \in \Delta$.

In 1977 [17] Kwong proved the following theorems:

Theorem 3.4.2 **(Kwong [22])**

If $a = 0$ and $b = \infty$, then conditional Dirichlet implies strong limit-point property at ∞ .

Theorem 3.4.3 **(Kwong [23])**

Let $a > -\infty$ and $b < \infty$. If $p^{-1} \notin L(a,b)$, then conditional Dirichlet at b implies strong limit-point property at b .

Theorem 3.4.4 **(Kwong [22])**

Let $b < \infty$, $p^{-1} \in L(a,b)$. If $\lim_{x \to b^-} \int_a^x q\,dx$ does not exist finitely, then conditional Dirichlet property at b implies strong limit-point at b .

In 1977 [23] Kwong has given an example in which $p^{-1} \in L(a,b)$ and $q \notin L(a,b)$ but $\lim_{x \to b} \int_a^x q$ exists, while M is limit-circle and conditional Dirichlet at the same time at b. This proves that $q \notin L(a,b)$ is not sufficient to ensure CD implies SLP.

In 1975 [13] Everitt proved that in Theorem 3.2.1 M(.) is satisfies strong limit-point and conditional Dirichlet at ∞ .

In 1976 [8] Evans proved the theorem when q on the form $q = q_1 - q_2$.

Theorem 3.4.5 **(Evans [8])**

Suppose that there exists a function $Q \in L_{loc}[0, \infty)$, a positive function $w \in AC_{loc}[0, \infty)$ and positive constants δ, k_1, k_2, k_3, such that :

(A) $q_1(x) \geq (1 + \delta) \dfrac{H^2(x)}{p(x)} - k_1/w^2(x)$, $H = \int Q$

(B) $\int_J (q_2 - Q)w \leq k_2 \inf_I p^{\frac{1}{2}}$ whenever $\int_I p^{-\frac{1}{2}} w^{-1} \leq 1$ and $J \subseteq I$.

(C) $p^{\frac{1}{2}} |w'| \leq k_3$

(D) $\int_0^\infty p^{-\frac{1}{2}} w = \infty$.

Then we have :

(i) M is limit-point at ∞ .

(ii) If w is bounded, $f \in \Delta \Longrightarrow wp^{\frac{1}{2}}f'$ and $w|q_1|^{\frac{1}{2}}f \in L^2(0,\infty)$

(iii) If w is bounded and there exist positive constants k_4 and k_5 such that

(E) either $|H(x)| < k_4 x$ and

$x < k_5 \; P^{\frac{1}{2}}(x) \, |q_1(x)|^{\frac{1}{2}} \, w^2(x)$,

or $Q = 0$, then for $f \in \Delta$

$$\lim_{x\to\infty} \int_0^x Q w^2 f^2 \quad , \quad \lim_{x\to\infty} \int_0^x (q_2 - Q) w^2 f^2$$

exist and are finite.

If $w = 1$, M(.) is conditional Dirichlet and strong limit-point at ∞ .

(iv) If Q is of one sign, $w = 1$ and instead of (B)

(B') $\int_I |q_2 - Q| \leq K_2 \inf_I p^{\frac{1}{2}}$ whenever $\int_I p^{-\frac{1}{2}} \leq 1$

then M(.) is Dirichlet at ∞ .

For the differential expression

$$M(y) = w^{-1} \left\{ -(py')' + qy \right\} \tag{2.5.27}$$

where p,q satisfy (3.1.1), $w(x) > o$ a.e. in [a,b) and $w \in L_{loc}$.
Amos in 1978 [2] proved the following :

Theorem 3.4.6 **(Amos [2])**

Let $-\infty < a < b = \infty$. Let P^{-1}, q; $w \in L(a, b)$

Then M(•) satisfies the Dirichlet condition and is in the limit-circle case in $L^2_w(a,\infty)$ at ∞ .

In 1985 [23] Race proved the following theorem for complex valued coefficients p and q .

Theorem 3.4.6 **(Race [23])**

Let $\theta \in [0, 2\Pi)$ and $\alpha \in (0, \Pi/2]$ be constants and suppose p take values in the sector defined by :

$$-\frac{\Pi}{2} + \alpha \leq \arg e^{i\theta} \; p(x) \leq \frac{\Pi}{2} - \alpha, \text{a.e } x \in [a,b)$$

Suppose also that p and w satisfy either

(i) $w \notin L(a,b)$ and $\int_a^b |P(x)|^{-1} \left(\int_a^x w \right) dx = \infty$

or

(ii) $w \in L(a,b)$ and $\int_a^b |P(x)|^{-1} \left(\int_x^b w \right) dx = \infty$

Then M is strong limit-point at b if and only if M is weak Dirichlet at b .

When $w = 1$

(i)' $\int_a^b (b - x)p^{-1} \, dx = \infty$ if $b < \infty$

(ii)' $\int_a^\infty x \, |p^{-1}(x)| \, dx = \infty$ if $b = \infty$.

Everitt in 1986 [16] gave sufficient conditions for the equation (2.5.27) to be Dirichlet and strong limit-point case at b .

§3.5 Application

In this section we shall deal with the cases in which the second-order linear differential expression M(.) satisfies the strong limit-point and Conditional Dirichlet properties.

Let M(.) be given by

$$M(f) = -f'' + qf \quad ; \quad (' \equiv \frac{d}{dx}) \text{ on } [0, \infty) \qquad (3.5.1)$$

where q is on the form

$$q(X) = aX^2 + b(X \cos X + \sin X) \qquad (3.5.2)$$

Where a and b are positive real numbers.

Let A be non-negative real number, and let ε , δ , τ be positive real numbers such that,

(i) $$b \left| \int_0^X \{ x \cos(x) + \sin(x) \} dx \right| \leq AX \qquad (3.5.3)$$

(ii) $$aX^2 \geq (1+\delta) b^2 \left[\int_0^X \{ x \cos(x) + \sin(x) \} dx \right]^2 - \varepsilon\tau X^2 \qquad (3.5.4)$$

Then

(i) M(.) is strong limit-point at ∞ .

(ii) M(.) is Conditional Dirichlet at ∞ .

For the proof of this result .

Let $\Delta \subset L^2(0,\infty)$ be the real manifold defined by :

$f \in \Delta$ if

(i) $f \in L^2(0,\infty)$

(ii) $f' \in AC_{loc}[0,\infty)$

(iii) $M(f) \in L^2(0,\infty)$.

Let $\Delta_0 \subset \Delta$ be defined by :

$$f \in \Delta_0 \quad \text{if} \quad f(0) = 0 \qquad (3.5.5)$$

Then for $f, g \in \Delta_0$; we have

$$f'g' + q\,f\,g = (f'g)' + gM(f)$$

Now, if $X > 0$, then we have

$$\int_0^X \{ f'g' + qfg \}\, d\bar{x} = (f'g)(X) + \int_0^X gM(f)\, dx \qquad (3.5.6)$$

Using (3.5.2) in (3.5.6) we get

$$\int_0^X \{f'g' + ax^2 fg\}\, dx = (f'g)(X) - b\int_0^X (x\cos x + \sin x) fg\, dx + \int_0^X g\, M(f)\, dx \qquad (3.5.7)$$

Consequently we have :

$$\int_0^X \{ f'^2 + ax^2 f^2 \} dx = (f'f)(X) - b\, f^2(X) \int_0^X (x\cos(x) + \sin(x))dx +$$

$$2b \int_0^X [\int_0^x (t\cos(t) + \sin(t))dt] ff'\, dx + \int_0^X fM(f)dx$$

$$\leq (f'f)(X) - bX\sin(X)\, f^2(X) + \tau \int_0^X f'^2 dx +$$

$$\tau^{-1} b^2 \int_0^X x^2 \sin^2(x)\, f^2(x)\, dx + \int_0^X fM(f)dx$$

where τ is a positive real number.

Hence

$$(1-\tau)\int_0^X f'^2(x)dx + a\int_0^X x^2 f^2(x)dx - \tau^{-1} b^2 \int_0^X x^2 \sin^2(x) f^2(x)dx$$

$$\leq (f'f)(X) + AX\, f^2(X) + \int_0^X fM(f)dx.$$

Now, if τ is in the form: $\tau = (1+\delta)^{-1/2}$, then $0<\tau<1$

and we have

$$(1-\tau)\int_0^X \{ f'^2(x) + ax^2 f^2(x) \} dx + \int_0^X \{ \tau a x^2 - \tau^{-1} b^2 x^2 \sin^2(x) - \varepsilon\tau x^2 \} f^2 dx$$

$$+ \tau\varepsilon \int_0^X x^2 f^2(x)dx \leq$$

$$(ff')(X) + AXf^2(X) + \int_0^X fM(f)\, dx.$$

which means that

$$(1-\tau)\int_0^X \{ f'^2(x) + ax^2f^2(x) \} dx + \varepsilon\tau\int_0^X x^2 f^2(x)\, dx$$

$$\leq (ff')(X) + AX f^2(X) + \int_0^X f\, M(f)\, dx$$

If $0 \leq X \leq t$, then we have

$$(1-\tau)\int_0^t dX \int_0^X \{ f'^2(x) + ax^2f^2(x) \} dx + \tau\varepsilon\int_0^t dX \int_0^X x^2f^2(x)dx$$

$$\leq \tfrac{1}{2} f^2(t) + A\int_0^t X^2f^2(X)dX + \int_0^t dX \int_0^X fM(f)dx.$$

i.e. $$(1-\tau)\int_0^t (t-X)\{ f'^2(X)+aX^2f^2(X) \} dX + \tau\varepsilon\int_0^t (t-X)X^2f^2(X)dX$$

$$\leq \tfrac{1}{2} f^2(t) + A\int_0^t X f^2(X)dX + \int_0^t (t-X)fM(f)dX.$$

This leads to

$$(1-\tau)\int_0^X (1-\tfrac{x}{X}) \{ f'^2(x) + ax^2f^2(x) \} dx + \tau\varepsilon\int_0^X (1-\tfrac{x}{X})x^2f^2(x)dx$$

$$\leq \tfrac{1}{2} X^{-1}f^2(X) + AX^{-1}\int_0^X x f^2(x)dx + \int_0^X (1-\tfrac{x}{X})fM(f)dx \quad (3.5.8)$$

From results, (4.1), (4.2) and (4.3) of [13] we get from (3.5.8) that :

(i) $f' \in L^2(0,\infty)$ (3.5.9)

(ii) $Xf \in L^2(0,\infty)$

By (3.5.7) it follows that,

$$\lim_{X \to \infty} (f'g)(X) \quad \text{exists and is finite} \qquad (3.5.10)$$

If this limit is not zero, then for some $\mu > 0$ and $X_0 > 0$,

$|f'g| \geq \mu$ on $[X_0, \infty)$.

Since f', $g \in L^2(0,\infty)$, then $fg' \in L(0,\infty)$, and this contradicts (3.5.10). Thus the limit is zero and M(.) is strong limit-point at ∞.

From (3.5.7) and (3.5.9) we have :

$$\lim_{X \to \infty} \int_0^X (x\cos(x) + \sin(x))\, f(x)g(x)dx \text{ exists and is}$$

finite and hence M(.) is Conditional Dirichlet at ∞.