CHAPTER IV

CHAPTER (IV)

Distribution of a complex of lines in Euclidean Space E^3 into one parametric family of normal congruences.

4.1 Points of orthogonality :

In this section we discuss the distribution of any arbitrary complex in E^3, into one parametric family of normal congruences. For this purpose we start with the definition of points of orthogonality. The differential equation of the complex of lines generated by $\underline{e}_3$, with respect to arbitrary frame, is given by (3.5) in the form :

$$w^2 = k\, w_3^1 + a\, w_3^2 + b\, w^1 . \qquad (4.1)$$

Let $\underline{M} = \underline{A} + t\, \underline{e}_3$ be any point on the ray of the complex which discribes an integral surface, orthogonal to the ray $\underline{e}_3$ of the complex. In this case we have :

$$w^3 + dt = 0 , \quad \text{with} \quad D\,(w^3 + dt) = 0 \qquad (4.2)$$

For the complex which is a three parametric family of lines, all the forms w^i, w_j^i are functions of u,v,w , du, dv, dw. The relation w = f(u,v) determines a congruence as a two parametric family of lines. Consider the relation

$$w^1 = A\, w_3^1 + B\, w_3^2 , \qquad (4.3)$$

which determines a congruence of lines of the complex (4.1). Exterior differentiation of (4.3) and using Cartan's lemma, we get :

$$[\Delta A\, w_3^1] + [\Delta B\, w_3^2] = \alpha \quad \text{where} ,$$

$$\Delta A = dA + (bA + k + \beta)\, w_2^1 - w^3$$

$$\Delta B = dB + (a + bB - A)\, w_2^1 \qquad (4.4)$$

Applying the exterior differentiation on $w^3 + dt = 0$ and substituting for w^2, w^1 from (4.1) and (4.3) we obtain :

$$B = k + bA . \qquad (4.5)$$

Thus the solution of the system

$$\left.\begin{aligned} w^3 + dt &= 0 \\ w^1 &= A\, w_3^1 + B\, w_3^2 \end{aligned}\right\} , \qquad (4.6)$$

exists by one arbitrary function of one variable i.e., any arbitrary complex (4.1) allows a distribution of normal congruences with orthogonal surface determined by one arbitrary function of one variable. The differential $d\,\underline{M}$ is :

$$d\,\underline{M} = (w^1 + tw_3^1)\,\underline{e}_1 + (w^2 + tw_3^2)\,\underline{e}_2 \qquad (4.7)$$

Put

$$w^1 + t\,w_3^1 = \bar{w}^1 , \quad w^2 + t\,w_3^2 = \bar{w}^2 ,$$

So

$$w^1 = \bar{w}^1 - tw_3^1 , \quad w^2 = \bar{w}^2 - tw_3^2 . \qquad (4.9)$$

From (4.1) and (4.3), (4.8) takes the form

$$w_3^1 = \alpha\,\bar{w}^1 + \beta\,\bar{w}^2 , \quad w_3^2 = \beta\,\bar{w}^1 + \gamma\,\bar{w}^2 , \qquad (4.10)$$

$$\alpha = \frac{1}{(\Lambda+t) - \dfrac{k+b\Lambda}{a+t}(k-tb)} + \frac{b}{\dfrac{a+t}{k+b\Lambda}(\Lambda+t)-(k-tb)},$$

$$\beta = -\frac{1}{\dfrac{(a+t)}{k+b\Lambda}(\Lambda+t) - (k-tb)} \qquad (4.11)$$

$$\gamma = \frac{1}{-\dfrac{k+b\Lambda}{\Lambda+t}(k-tb) + (a+t)} .$$

The Gaussian curvature K of the surface is determined by

[11] $$K = \alpha\gamma - \beta^2$$

Using (4.11), takes the form

$$K = \frac{1}{(a+t)(\Lambda+t) - (k+\Lambda b)(k-tb)} \qquad (4.12)$$

The expression for Λ is

$$\Lambda = \frac{1 + K k^2 - K t^2 - aKt - bKkt}{K(b^2 t + a - bk + t)} .$$

Differentiating (4.13) we get

$d\Lambda =$

$$= \frac{1}{K} \frac{[\Delta k + (bk-a)w_2^1 - bw^2] - w^2 + [\Delta a + (ab+k)w_2^1 + w^3] + [\Delta b + (1+b^2)w_2^1]\mu}{\delta^2}$$

(4.14)

where

$$\xi = 2b^2Kkt + 2aKb - bKb^2 + 2Kkt - b^3Kt^2 - 2abKt - 2bKt^2 + b .$$

$$\xi = b^2Kt^2 - 2aKt + 2bKkt - Kt^2 - a^2K - b^2 - Kk^2 - 1 ,$$

$$\lambda = -b^2Kt^2 + 2bKkt - Kk^2 - 1 , \qquad (4.15)$$

$$\mu = -2aKkt - 2Kkt^2 - 2bt - 2bKk^2t + 2bKt^3 + 2abkt^2 + \\ + b^2Kkt^2 + Kk^2 + k$$

$$\delta^2 = b^4t^2 + a^2 + b^2k^2 + t^2 + 2ab^2t - 2b^3kt + 2b^2t^2 - 2abk + \\ + 2\,at - 2bkt.$$

Introducing the values of ΔA and ΔB given by (4.4), we get :

$$[dA + 2(k + bA)\overset{1}{w}_2 - \overset{3}{w}\overset{2}{w}_3] + [db + (a+bk+b^2A-A)\overset{1}{w}_2\overset{2}{w}_3] = 0 \qquad (4.16)$$

By virtue of (4.14),(4.15) and (3.8), we have

$$-[\xi(\alpha K\delta k + \alpha ba' + \gamma K\delta) + \lambda(\beta K\delta k + \beta ba' + K\delta r) + \mu(pK\delta k + pba' + K\delta\beta)]$$

$$+(K\delta^2 + b\xi)(\alpha a' + qK\delta) + b\lambda(\beta a' + \gamma K\delta) + (\delta a' + b\mu)(pa' + \alpha K\delta) = 0 \qquad (4.17)$$

where ;

$$\delta = b^2t + a - bk + t ,$$
$$a' = b^2t + a - bk + t + 2b^2Kk^2t + aKb^2 - bKk^3 + Kb^2t - \\ - b^2Kt^3 - 2aKt^2 - Kt^3 - ab^2Kt^2 - a^2Kt - b^3Kkt^2 . \qquad (4.18)$$

Removing brakets in (4.16) we obtain an equation of the fourth order in t, in the form :

$$pKt^4 - 2Kt^3(\alpha + b\beta - ap) + t^2(Kq - 2p + 2k\beta K - 4aK\alpha + b^4Kq + 2b^2Kq + 2bK\gamma - 2abK\beta + a^3Kr$$
$$b^2kr) - 2t(Kk\gamma - \alpha + a^2K\alpha - ab^2Kq + b^3Kkq - aKq + bKkq - abK\gamma + bKkr - aKk\beta - b\beta + ap) +$$
$$+ \frac{1}{K}(p + Kr)(1 + k^2K) + 2a\alpha - 2bk\alpha + a^2Kq + b^2Kk^2q - 2abKkq - 2b\gamma - 2aKk\gamma = 0 . \qquad (4.19)$$

In canonical frame ($a = b = 0$) equation (4.19) takes the form :

$$pKt^4 - 2\alpha Kt^3 + t^2(Kq - 2p + 2k\beta K) - 2t(k\gamma K - \alpha) + \frac{1}{K}(p+Kr)(1+k^2K) = 0 \qquad (4.20)$$

This equation gives abssica of the point $\underline{M} = \underline{A} + t\,\underline{e}_3$ which discribes a surface cutting orthogonolly the rays of the normal congruence. It is clear that on the ray of the complex there are four points each of them describes a surface cutting orthogonally the rays of the normal congruence. These points are called points of orthogonality.

4.2 The investigation of particular kinds of complexes of lines in E^3 which are distributed into special kinds of normal congruences.

We investigate, the complex of lines which has the following property :

Every ray of the complex has a double points of orthogonality coincide with the point at infinity ($t = \infty$).

From the equation which gives the points of orthogonality,

$$pKt^4 - 2\alpha Kt^3 + t^2(Kq - 2p + 2k\beta K) - 2t(k\gamma K - \alpha) + \frac{1}{K}(p+Kr)(1+k^2K) = 0,$$

if $t = \infty$ is a double root of the above equation, then

$$p = \alpha = 0 . \qquad (4.21)$$

In this case [using equations (3.9)'] we have the following system of differential equations :

$\omega^1 = k\,\omega_3^2 - \frac{r}{\beta}\,\omega_3^2 - \frac{\gamma}{\beta}\,\omega_3^1$

$$
\begin{aligned}
& w^2 = k w_3^1 \;, \\
& w_1^2 = \beta w_3^2 \;, \\
& dk = q\, w_3^1 + \gamma\, w_3^2 \;, \\
& -w^3 + k w_1^2 = \beta w^1 + \gamma w_3^1 + r w_3^2 \;, \quad (w^3 = (k\beta - r) w_3^2 - \beta w^1 - \gamma w_3^1) \;.
\end{aligned}
\tag{4.22}
$$

If the complex (4.22) is distributed into one parametric family of congruences

$$w^1 = a\, w_3^1 + k\, w_3^2 \;, \tag{4.23}$$

with rays cutting orthogonaly the surface

$$w^3 = 0 \;. \tag{4.24}$$

then, from (3.9)' the equation $w^3 = 0$ gives :

$$w^1 = \frac{k\beta - r}{\beta}\, w_3^2 - \frac{\gamma}{\beta}\, w_3^1 \;, \tag{4.25}$$

which is identified with (4.23) i.e., we have

$$a = -\frac{\gamma}{\beta} \;, \qquad r = 0 \;.$$

Hence the differential equations. of the complex, characterized by $p = \alpha = r = 0$, are :

$$
\begin{aligned}
& w^2 = k\, w_3^1 \;, \\
& w_1^2 = \beta\, w_3^2 \;, \\
& dk = q\, w_3^1 + \gamma\, w_3^2 \;, \\
& -w^3 + k w_1^2 = \beta w^1 + \gamma w_3^1 \;, \quad (w^3 = -\beta w^1 - \gamma w_3^1 + k\beta\, w_3^2) \;.
\end{aligned}
\tag{4.26}
$$

Exterior differentiation of the last three equations gives :

$$[d\beta + (1+\beta^2)\overset{1}{w}_3 \overset{2}{w}_3] = 0 \quad ,$$
$$[dq\ \overset{1}{w}_3] + [d\gamma + \gamma\beta \overset{1}{w}_3 \overset{2}{w}_3] = 0 \quad , \qquad (4.27)$$
$$[d\beta + (1+\beta^2)\overset{1}{w}_3 \overset{1}{w}] + [d\gamma + \gamma\beta \overset{1}{w}_3 - (2k\beta^2 + q\beta)\overset{1}{w}_3 \overset{2}{w}_3] = 0 \ .$$

Removing brackets, by virtue of Cartan's lemma, in equations (4.27) , we obtain :

$$d\beta + (1+\beta^2)\overset{1}{w}_2 = \lambda_1 \overset{2}{w}_3 \quad ,$$
$$dq = \lambda_2 \overset{1}{w}_3 + \lambda_3 \overset{2}{w}_3 \quad ,$$
$$d\gamma + \gamma\beta\overset{1}{w}_3 = \lambda_3 \overset{1}{w}_3 + \lambda_4 \overset{2}{w}_3 \ , \qquad (4.28)$$
$$d\beta + (1+\beta^2)\overset{1}{w}_3 = \lambda_5 \overset{1}{w} + \lambda_6 \overset{2}{w}_3 \quad ,$$
$$d\gamma + \gamma\beta\overset{1}{w}_3 - \beta(q+2k\beta)\overset{1}{w}_3 = \lambda_6 \overset{1}{w} + \lambda_7 \overset{2}{w}_3 \ .$$

Comparing the values of $d\beta$ and $d\gamma$ in (4.31), so

$$\lambda_1 = \lambda_5 = \lambda_6 = 0 \ ,$$
$$\lambda_3 = \beta(q + 2k\beta) \quad ,$$
$$\lambda_4 = \lambda_7 \ .$$

By virtue of Cartan's common method for the systems (4.26) and (4.28), it follows that there are only two independent parameters i.e., $N = 2$: (λ_2, λ_4). The number of independent quadratic exterior forms is $S_1 = 2$. The number of independent characteristic