Using the adjoint representations [5], in Table (2), we can find thirty-one different kinds of solutions corresponding to the basic fields of an optimal system given by

$$\chi_1 + c\chi_2,\quad \chi_1 + c\chi_3,\quad \chi_2 + c\chi_3,\quad \chi_1 + \chi_2 + c\chi_3,\quad \chi_2 + \chi_4 - \frac{(1+\gamma^2)}{\gamma}\chi_{10},$$

$$\chi_1 + \chi_4 - \frac{(1+\gamma^2)}{\gamma}\chi_{10},\quad \chi_1 + \chi_4, \chi_2 + \chi_4,\quad \chi_2 + \chi_5,\quad \chi_1 + \chi_{10},\quad \chi_2 + \chi_{10},$$

$$\chi_1 + \chi_7,\quad \chi_2 + \chi_7,\quad \chi_2 + \chi_6,\quad \chi_2 + \chi_{11},\quad \chi_2 + \chi_8,\quad \chi_2 + \chi_9 + \chi_{11}$$

$$\chi_1 + \chi_9 - \frac{k}{\gamma}\chi_{11},\quad \chi_1 + \chi_8,\quad \chi_1 + \chi_6,\quad \chi_1 + \chi_{11},\quad \chi_1 + \chi_2 + \chi_6$$

$$\chi_1 + \chi_2 + \chi_4 - \frac{(1+\gamma^2)}{\gamma}\chi_{10},\quad \chi_1 + \chi_2 + \chi_6 + \chi_{11},\quad \chi_1 + \chi_2 + a_1(\chi_9 - \frac{k}{\gamma}\chi_{11})$$

$$+ a_2\chi_8,\quad \chi_1 + \chi_2 + \chi_9 - \frac{(\gamma - k)}{\gamma}\chi_{11},\quad \chi_1 + \chi_5,\quad \chi_1 + \chi_{12},\quad \chi_2 + \chi_{12},$$

$$\chi_2 + \chi_8 + \frac{(1+\gamma^2)}{\gamma}\chi_{12},\quad \chi_1 + \chi_8 + \frac{(1+\gamma^2)}{\gamma}\chi_{12},$$

Now, we will integrate the following characteristic equations:

$$\frac{dx}{X} = \frac{dt}{T} = \frac{du}{U} = \frac{d\theta}{\Theta}. \qquad (3.42)$$

To get the similarity variable, p, we integrate the first equation $\frac{dx}{X} = \frac{dt}{T}$.

To get the similarity solutions for the reduced ordinary differential equation system $f(p)$ we integrate $\frac{dx}{X} = \frac{d\theta}{\Theta}$ and to get $h(p)$ we integrate $\frac{dx}{X} = \frac{du}{U}$.

Next, we shall make use of the symmetries χ_i to obtain sub algebras leading to nontrivial invariant solutions, as the following $(\chi_1, \chi_2), (\chi_1, \chi_3), \ldots$ each of them provides us with ansatz which reduce system (3.1a, 3.1b) to a system of ordinary differential equation. Some cases of sub-algebras will be considered. In the following the constants of integrations will be denoted by $a_1, a_2, \ldots$.

Case 1

For the sub-algebra (χ_1, χ_2), consider the linear combination of χ_1 and χ_2 by $[c\chi_1 + \chi_2]$. We introduce here the parameter c to demonstrate that not only will $[\chi_1 + \chi_2]$ give a specific similarity solution but also a linear combination of χ_1 and χ_2 with arbitrary coefficients. The group invariant combination suggests the similarity variables p and the similarity solutions $f(p)$ and $h(p)$ to be

$$p = cx - t, \quad u(x,t) = h(p) \text{ and } \theta(x,t) = f(p)$$

Substituting similarity solutions into system (3.1a, 3.1b), we have the following reduced system of ODE's

$$(1-c^2)h'' + \gamma c f' = 0, \tag{3.43a}$$

$$c\gamma h'' + f' + kc^2 f'' = 0. \tag{3.43b}$$

This system has exact solution

$$h(p) = \left[\frac{a_1}{(1-c^2)} - \frac{\gamma c a_2}{(1-c^2)(1-c^2-c^2\gamma^2)}\right]p + \left[\frac{\gamma k c^3 a_3}{(1-c^2-c^2\gamma^2)}\right]\exp\left(-\left(\frac{1-c^2-c^2\gamma^2}{kc^2(1-c^2)}\right)p\right) + a_4 \tag{3.44a}$$

and

$$f(p) = \frac{a_2}{(1-c^2-c^2\gamma^2)} + a_3 \exp\left(-\left(\frac{1-c^2-c^2\gamma^2}{kc^2(1-c^2)}\right)p\right). \tag{3.44b}$$

By back transformation, we get

$$u(x,t)=\left[\frac{a_1}{(1-c^2)}-\frac{\gamma c a_2}{(1-c^2)(1-c^2-c^2\gamma^2)}\right](cx-t)+$$
$$+\left[\frac{\gamma k c^3 a_3}{(1-c^2-c^2\gamma^2)}\right]\exp(-(\frac{1-c^2-c^2\gamma^2}{kc^2(1-c^2)})(cx-t))+a_4 \qquad (3.45a)$$

and

$$\theta(x,t)=\frac{a_2}{(1-c^2-c^2\gamma^2)}+a_3\exp(-(\frac{1-c^2-c^2\gamma^2}{kc^2(1-c^2)})(cx-t)). \qquad (3.45b)$$

<u>*Case 2*</u>

For the sub-algebra (χ_1,χ_3), consider the linear combination $[\chi_1+c\chi_3]$. The general reduction of this sub group can be obtained by the following similarity representation

$$p=t,\quad u(x,t)=h(p)\exp(cx) \text{ and } \theta(x,t)=f(p)\exp(cx),$$

which reduces the system (3.1a, 3.1b) to the following system of ODE 's

$$h''-c^2h+c\gamma f=0, \qquad (3.46a)$$

$$c\gamma h'+f'-c^2kf=0 \qquad (3.46b)$$

To solve this system, we have four cases:

1- For $\gamma=\sqrt{2k^2-\frac{1}{2}}$ and $c=1$ we have the following exact solution

$$h(p)=a_1\exp(2kp)+a_2\exp(m_1p)+a_3\exp(m_2p), \qquad (3.47a)$$

$$f(p)=\frac{1}{\gamma}[(1-4k^2)a_1\exp(2kp)+(1-m_1^2)a_2\exp(m_1p)$$
$$+(1-m_2^2)a_3\exp(m_2p)], \qquad (3.47b)$$

where $m_{1,2}=\frac{-k\pm\sqrt{k^2+2}}{2}$, then by back transformation we get

$$u(x,t)=[a_1\exp(2kt)+a_2\exp(m_1t)+a_3\exp(m_2t)]\exp(x), \qquad (3.48a)$$

$$\theta(x,t)=\frac{1}{\gamma}[(1-4k^2)a_1\exp(2kt)+(1-m_1^2)a_2\exp(m_1t) \\ +(1-m_2^2)a_3\exp(m_2t)]\exp(x) \tag{3.48b}$$

2- For $\gamma=\sqrt{3-\frac{3}{2}k}$ and $c=1$ we have the following exact solution

$$h(p)=a_1\exp(2p)+a_2\exp(m_1p)+a_3\exp(m_2p), \tag{3.49a}$$

$$f(p)=\frac{1}{\gamma}\left[(-3a_1\exp(2p)+(1-m_1^2)a_2\exp(m_1p)+(1-m_2^2)a_3\exp(m_2p)\right], \tag{3.49b}$$

where $m_{1,2}=\dfrac{(k-2)\pm\sqrt{(2-k)^2+2k}}{2}$, then using back transformation we get

$$u(x,t)=(a_1\exp(2t)+a_2\exp(m_1t)+a_3\exp(m_2t))\exp(x), \tag{3.50a}$$

$$\theta(x,t)=\frac{1}{\gamma}\left[(-3a_1\exp(2t)+(1-m_1^2)a_2\exp(m_1t)+(1-m_2^2)a_3\exp(m_2t)\right]\exp(x), \tag{3.50b}$$

3- For $\gamma=\sqrt{6k^2-\frac{3}{2}}$ and $c=1$ we have the following exact solution

$$h(p)=a_1\exp(-2kp)+a_2\exp(m_1p)+a_3\exp(m_2p), \tag{3.51a}$$

$$f(p)=\frac{1}{\gamma}[((1-4k^2)a_1\exp(-2kp)+(1-m_1^2)a_2\exp(m_1p) \\ +(1-m_2^2)a_3\exp(m_2p)], \tag{3.51b}$$

where $m_{1,2}=\dfrac{3k\pm\sqrt{9k^2-2}}{2}$,then by back transformation we get

$$u(x,t)=(a_1\exp(-2kt)+a_2\exp(m_1t)+a_3\exp(m_2t))\exp(x) \tag{3.52a}$$

$$\theta(x,t)=\frac{1}{\gamma}[(1-4k^2)a_1\exp(-2kt)+(1-m_1^2)a_2\exp(m_1t) \\ +(1-m_2^2)a_3\exp(m_2t)]\exp(x) \tag{3.52b}$$

4- For $\gamma = \sqrt{6k - \frac{3}{2}}$ and $c = \frac{1}{\sqrt{k}}$ we have exact solution

$$h(p) = a_1 \exp(-2p) + a_2 \exp(m_1 p) + a_3 \exp(m_2 p), \tag{3.53a}$$

$$f(p) = \frac{\sqrt{k}}{\gamma}[(\frac{1}{k} - 4)a_1 \exp(-2p) + (\frac{1}{k} - m_1^2)a_2 \exp(m_1 p) + (\frac{1}{k} - m_2^2)a_3 \exp(m_2 p)] \tag{3.53b}$$

where $m_{1,2} = \frac{3k \pm \sqrt{9k^2 - 2k}}{2k}$, then by back transformation we get

$$u(x,t) = (a_1 \exp(-2t) + a_2 \exp(m_1 t) + a_3 \exp(m_2 t)) \exp(\frac{x}{\sqrt{k}}), \tag{3.54a}$$

$$\theta(x,t) = \frac{\sqrt{k}}{\gamma}[(\frac{1}{k} - 4)a_1 \exp(-2t) + (\frac{1}{k} - m_1^2)a_2 \exp(m_1 t) + (\frac{1}{k} - m_2^2)a_3 \exp(m_2 t)] \exp(\frac{x}{\sqrt{k}}) \tag{3.54b}$$

Case 3

For the sub-algebra (χ_2, χ_3), consider the linear combination $[\chi_2 + c\chi_3]$. The general reduction of this sub group can be obtained by the following similarity representation

$$p = x, \quad u(x,t) = h(p)\exp(ct), \quad \theta(x,t) = f(p)\exp(ct),$$

Substituting similarity solutions into system (3.1a, 3.1b), we have the following system of ODE's

$$c^2 h - h'' + \gamma f' = 0, \tag{3.55a}$$

$$c\gamma h' + cf - kf'' = 0. \tag{3.55b}$$

We will investigate three cases, where *A* and *B* are arbitrary constants.

1- For $c=-k$ and $\gamma=\sqrt{k^2-1}$, The exact solution for (3.55a, 3.55b) is

$$h(p)=A\sinh(\sqrt{k}\,p)+B\cosh(\sqrt{k}\,p), \qquad (3.56a)$$

$$f(p)=\frac{1}{\gamma}\left[(1-k)\sqrt{k}A\cosh(\sqrt{k}\,p)+(1-k)\sqrt{k}B\sinh(\sqrt{k}\,p)\right], \qquad (3.56b)$$

By back transformation, we get

$$u(x,t)=[A\sinh(\sqrt{k}x)+B\cosh(\sqrt{k}x)]\exp(-kt), \qquad (3.57a)$$

$$\theta(x,t)=\frac{1}{\gamma}\left[(1-k)\sqrt{k}A\cosh(\sqrt{k}x)+(1-k)\sqrt{k}B\sinh(\sqrt{k}x)\right]\exp(-kt), \qquad (3.57b)$$

2- For $k=\gamma=1$ and $c=1$ The solution of (3.55a, 3.55b) is

$$h(p)=A\cos(\sqrt{\frac{1+\sqrt{5}}{2}}p), \qquad (3.58a)$$

and

$$f(p)=-A(\frac{3+\sqrt{5}}{\sqrt{2(1+\sqrt{5})}})\sin(\sqrt{\frac{1+\sqrt{5}}{2}}p), \qquad (3.58b)$$

By back transformation, we get

$$u(x,t)=A\cos(\sqrt{\frac{1+\sqrt{5}}{2}}x)\exp(-t), \qquad (3.59a)$$

$$\theta(x,t)=-A(\frac{3+\sqrt{5}}{\sqrt{2(1+\sqrt{5})}})\sin(\sqrt{\frac{1+\sqrt{5}}{2}}x)\exp(-t). \qquad (3.59b)$$

3- For $k=\gamma=1$ and $c=-1$ the solution of this equation is

$$h(p)=A\sinh(\sqrt{\frac{-1+\sqrt{5}}{2}}p), \qquad (3.60a)$$

and

$$f(p)=A(\frac{-3+\sqrt{5}}{\sqrt{2(-1+\sqrt{5})}})\cosh(\sqrt{\frac{-1+\sqrt{5}}{2}}p), \qquad (3.60b)$$