PREFACE

PREFACE

The matrix exponential plays a central role in linear systems and control theory. Mathematical models of many physical, biological, and economic processes involve systems of linear, constant coefficient ordinary differential equations $\dot{x}(t) = A\,x(t)$ where A is a given fixed, real or complex $n \times n$ matrix. The solution to this equation is given by

$x\ (t)\ = e^{At}\, x(0)$, where $e^{At} = \sum_{k=0}^{\infty} \frac{(At)^k}{k!}$ denotes the exponential of the matrix At and $x(0)$ is the initial solution. In this thesis :

***We** give explicit formulas for computing the exponential of some special matrices and of some special block matrices.

***Also** , we present the use and modification of a recent technique , so called restrictive Padé approximation , used to approximate the exponential matrix and compute the transient response vectors for the above system .

***Also ,** we consider the nonlinear matrix equation

$$X - A^{*} F(X) A = I \quad , \qquad F(X) = \sqrt{X^{-1}} \, .$$

This type of matrix equation often arises in the analysis of ladder, networks, optimal control theory, dynamic programming, stochastic filtering, and statistics. The iteration process $X_{k+1} = I + A^{*} F(X_k) A$, $k=0,1,2,\ldots$ is investigated (under some conditions) to obtain the positive definite solution for this equation . Some numerical examples which describe the performance of the algorithm are presented .

This thesis consists of five chapters where the first two chapters begin with an introduction, in which are given some information and ideas on its contents .

Chapter one :

We summarize some basic concepts, definitions and theorems, for computing the exponential matrix e^A .

Chapter two :

In this chapter we will introduce the linear and nonlinear matrix equations, we present the Lyapunov matrix equation as an example to the linear matrix equation . Also, some properties of the positive definite solution to the nonlinear matrix equation $X+A^T X^{-1} A = I$ are discussed , where the smallest and largest positive definite solutions are mentioned.

Chapter three :

In this chapter, we introduce some explicit formulas for some special types of block matrices that appear very often in control theory. Such block matrices are related to the second-order mechanical vibration equation $M\ddot{x} + C\dot{x} + Kx = 0$ where M, C and K are real or complex $n \times n$ matrices.

Chapter four :

In this chapter, a hybrid technique is presented to compute the transient response vectors in such an iterative explicit form . A recent method , so called restrictive Padé approximation , is used and extended to approximate the exponential matrix [14] . Numerical test example is given to illustrate the accuracy of the suggested method compared with previous works as well as the exact solution.

The computations were done using MATLAB package version 5.1 where long E format were used.

Chapter five :

In this chapter, we establish and prove theorems for the existence of the iteration solution of the nonlinear matrix equation $X - A^{*}\sqrt{X^{-1}}A = I$, $A, X \in P(n)$ where $P(n)$ denotes the set of all positive definite $n \times n$ matrices. Also we obtain the rate of convergence for the sequence. Some numerical examples are given to illustrate the performance of the algorithm. The algorithm is programmed using (Turbo C++ version 3.0).

CHAPTER (1)

BASIC CONCEPTS OF NUMERICAL LINEAR ALGEBRA AND MATRIX COMPUTATION

CHAPTER (1)

BASIC CONCEPTS OF NUMERICAL LINEAR ALGEBRA AND MATRIX COMPUTATION

In this chapter we are going to give some basic concepts, definitions and theorems about numerical linear algebra and matrices. These concepts are needed in the next chapters.

1.1 : TERMINOLOGY AND DEFINITIONS

Definition : 1.1.1

The characteristic polynomial of a square matrix A of order n is $\det(A-xI)=0$, where x denotes the eigenvalues of A .

Theorem : 1.1.2 (Cayley-Hamilton Theorem) [12]

Every square matrix satisfies its own characteristic equation. That is , if $\det(A-xI) = c_n x^n + c_{n-1}x^{n-1} + \ldots + c_2x^2 + c_1x + c_o$ then

$$c_n A^n + c_{n-1}A^{n-1} + \ldots + c_2A^2 + c_1A + c_oI = 0 \ .$$

Example :

For the matrix $A = \begin{pmatrix} 5 & 2 & 2 \\ 3 & 6 & 3 \\ 6 & 6 & 9 \end{pmatrix}$

its characteristic equation is $-x^3+ 20x^2-93x+126= 0$ where x denotes the eigenvalues of A . Therefore, we have $-A^3 + 20A^2 - 93A + 126I = 0$.

REMARK :

From the Cayley-Hamilton theorem we find that every well-defined function of an $n\times n$ matrix A can be expressed as a polynomial of degree $(n-1)$ in A. Thus,

$$f(A)=a_0I+a_1A+a_2A^2+...+a_{n-1}A^{n-1} \qquad (1.1.1)$$

where the scalars $a_o, a_1, a_2, \ldots, a_{n-1}$ are determined as follows :

STEP 1 :

Let $$g(\lambda)=a_0+a_1\lambda+a_2\lambda^2+...+a_{n-1}\lambda^{n-1}$$

which is the right side of (1.1.1) with A^i replaced by λ^i $i=(0,1, \ldots, n-1)$.

STEP 2 :

For each distinct eigenvalues λ_i of A, formulate the equation

$$f(\lambda_i)=g(\lambda_i) \qquad (1.1.2)$$

STEP 3 :

If λ_i is an eigenvalue of multiplicity k, for $k>1$, then formulate also the following equations, involving derivatives of $f(\lambda)$ and $g(\lambda)$ with respect to λ :

$$\begin{aligned} f'(\lambda)\Big|_{\lambda=\lambda_i} &= g'(\lambda)\Big|_{\lambda=\lambda_i} \\ f''(\lambda)\Big|_{\lambda=\lambda_i} &= g''(\lambda)\Big|_{\lambda=\lambda_i} \qquad (1.1.3) \\ \ldots\ldots\ldots \ \ldots\ldots\ldots \ &\ldots\ldots\ldots \ \ldots\ldots \\ f^{(k-1)}(\lambda)\Big|_{\lambda=\lambda_i} &= g^{(k-1)}(\lambda)\Big|_{\lambda=\lambda_i} \end{aligned}$$

STEP 4 :

Solve the set of all equations obtained in steps (1.1.2) and (1.1.3) for the unknown scalars $a_o, a_1, a_2, \ldots, a_{n-1}$, then by substituting into (1.1.1) by these scalars, $f(A)$ may be calculated.

Example : To find e^A for the matrix $A=\begin{pmatrix} 2 & 4 \\ 1 & 2 \end{pmatrix}$

from (1.1.1) we can write

$$e^A = a_o I + a_1 A = \begin{pmatrix} a_0 + 2a_1 & 4a_1 \\ a_1 & a_o + 2a_1 \end{pmatrix} \tag{1.1.4}$$

now , $f(\lambda) = e^{\lambda}$, $g(\lambda) = a_o + a_1\lambda$ and the distinct eigenvalues of A are $\lambda_1 = 0$, $\lambda_2 = 4$ then from (1.1.2) for each eigenvalue, we formulate the two equations $e^0 = a_0 + 0(a_1)$ and $e^4 = a_0 + 4(a_1)$.

By solving this system for a_0, a_1 we have $a_o = 1$, $a_1 = 13.399$ and by substituting of these values into (1.1.4) gives us

$$e^A = \begin{pmatrix} 27\ .7991 & 53.5982 \\ 13.3995 & 27.7991 \end{pmatrix} .$$

Definition : 1.1.3

The $n \times n$ matrix C of the form

$$\begin{bmatrix} 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & & \cdot & \cdot \\ \cdot & \cdot & \cdot & & \cdot & \cdot \\ 0 & 0 & 0 & 0 & \cdots & 1 \\ -c_o & -c_1 & -c_2 & -c_3 & \cdots & -c_{n-1} \end{bmatrix}$$

is said to be the companion matrix of the polynomial

$$f(x) \equiv x^n + c_{n-1}x^{n-1} + c_{n-2}x^{n-2} + \ldots + c_1 x + c_0 .$$

SIMILARITY AND JORDAN CANONICAL FORM

A matrix A is similar to matrix B if there exists an invertible matrix S such that $A = S^{-1}B\,S$. If A is similar to a matrix B, then B is similar to the matrix A and both matrices must be of the same order and square. Similar matrices have the same characteristic equation ,therefore the same eigenvalues .

Definition : 1.1.4

A Jordan block is a square matrix whose diagonal elements are all equal , whose superdiagonal elements are all equal 1, and whose other elements are all zero. That is it has the form :

$$\begin{pmatrix} \lambda & 1 & 0 & \dots & 0 & 0 \\ 0 & \lambda & 1 & \dots & 0 & 0 \\ 0 & 0 & \lambda & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & \lambda & 1 \\ 0 & 0 & 0 & \dots & 0 & \lambda \end{pmatrix}$$

A Jordan block is completely determined by its order and the value of its diagonal elements. Now, we can define the Jordan canonical form as follows :

A matrix is in Jordan canonical form if it is a diagonal matrix or if it has one of the following two partitioned forms

$$\begin{pmatrix} D & & & & & O \\ & J_1 & & & & \\ & & J_2 & & & \\ & & & \cdot & & \\ & & & & \cdot & \\ & O & & & & \cdot \\ & & & & & & J_r \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} J_1 & & & & O \\ & J_2 & & & \\ & & \cdot & & \\ & & & \cdot & \\ O & & & & \cdot \\ & & & & & J_r \end{pmatrix}$$

where D denotes a diagonal matrix and J_i (i=1,2,3,...,r) represents a Jordan block .

Definition : 1.1.5

If J is the $r \times r$ Jordan block

$$J = \begin{pmatrix} \lambda & 1 & 0 & \cdots & 0 & 0 \\ 0 & \lambda & 1 & \cdots & 0 & 0 \\ 0 & 0 & \lambda & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & \lambda & 1 \\ 0 & 0 & 0 & \cdots & 0 & \lambda \end{pmatrix}$$

then the function of J is defined by

$$f(J) = \begin{pmatrix} \frac{f(\lambda)}{0!} & \frac{f'(\lambda)}{1!} & \frac{f''(\lambda)}{2!} & \cdots & \frac{f^{(r-2)}(\lambda)}{(r-2)!} & \frac{f^{(r-1)}(\lambda)}{(r-1)!} \\ 0 & \frac{f(\lambda)}{0!} & \frac{f'(\lambda)}{1!} & \cdots & \frac{f^{(r-3)}(\lambda)}{(r-3)!} & \frac{f^{(r-2)}(\lambda)}{(r-2)!} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & \frac{f(\lambda)}{0!} & \frac{f'(\lambda)}{1!} \\ 0 & 0 & 0 & \cdots & 0 & \frac{f(\lambda)}{0!} \end{pmatrix}$$

where all derivatives are taken with respect to λ. Also , if J is the diagonal matrix

$$J = \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & \lambda_n \end{pmatrix} \quad \text{then} \quad f(J) = \begin{pmatrix} f(\lambda_1) & 0 & \cdots & 0 \\ 0 & f(\lambda_2) & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & f(\lambda_n) \end{pmatrix}$$

Finally , if A is similar to the matrix J in Jordan canonical form i.e $A = MJM^{-1}$, then $f(A) = M\,f(J)\,M^{-1}$, where M is a modal matrix for A.

Example :

To calculate e^J for the Jordan block

$$J = \begin{pmatrix} 2 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 2 \end{pmatrix}$$

we find $f(\lambda)=e^{\lambda}$, $f^{(1)}(\lambda)=f^{(2)}(\lambda)=f^{(3)}(\lambda)=e^{\lambda}$, so

$f(2)=e^2$, $f^{(1)}(2)=f^{(2)}(2)=f^{(3)}(2)=e^2$ then , we have

$$f(J)=\begin{pmatrix} \frac{e^2}{1} & \frac{e^2}{1} & \frac{e^2}{2} & \frac{e^2}{6} \\ 0 & \frac{e^2}{1} & \frac{e^2}{1} & \frac{e^2}{2} \\ 0 & 0 & \frac{e^2}{1} & \frac{e^2}{1} \\ 0 & 0 & 0 & \frac{e^2}{1} \end{pmatrix}$$

$$=\begin{pmatrix} 7.389056 & 7.389056 & 3.694528 & 1.231509 \\ 0 & 7.389056 & 7.389056 & 3.694528 \\ 0 & 0 & 7.389056 & 7.389056 \\ 0 & 0 & 0 & 7.389056 \end{pmatrix}$$

Example :

To calculate e^A for the matrix $A=\begin{pmatrix} 5 & 2 & 2 \\ 3 & 6 & 3 \\ 6 & 6 & 9 \end{pmatrix}$

By solving the cubic equation $|A-\beta I|=0$ in $\beta=\{\lambda_1, \lambda_2, \lambda_3\}$ where λ_i are eigenvalues of A , we have $\lambda_1=3, \lambda_2=3, \lambda_3=14$ also we can calculate the eigenvectors $[-1, 1, 0]^T$; $[-1, 0, 1]^T$; $[2, 3, 6]^T$ then , a modal matrix M and the diagonal matrix J are

$$M=\begin{pmatrix} -1 & -1 & 2 \\ 1 & 0 & 3 \\ 0 & 1 & 6 \end{pmatrix} \quad \text{and} \quad J=\begin{pmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 14 \end{pmatrix}$$

where A is written in the form $A=MJM^{-1}$, then

$$e^A = M e^J M^{-1}$$

$$= \begin{pmatrix} -1 & -1 & 2 \\ 1 & 0 & 3 \\ 0 & 1 & 6 \end{pmatrix} . \begin{pmatrix} e^3 & 0 & 0 \\ 0 & e^3 & 0 \\ 0 & 0 & e^{14} \end{pmatrix} . \begin{pmatrix} -3/11 & 8/11 & -3/11 \\ -6/11 & -6/11 & 5/11 \\ 1/11 & 1/11 & 1/11 \end{pmatrix}$$

$$= 10^5 \times \begin{pmatrix} 2.186717580147 & 2.186516724777 & 2.186516724777 \\ 3.279775087166 & 3.279975942536 & 3.279775087166 \\ 6.559550174333 & 6.559550174333 & 6.559751029702 \end{pmatrix} .$$

Definition : 1.1.6

Let $A \in C^{n\times n}$ and suppose that $A^2 = 0$, then this matrix is called nilpotent .

The matrix $A = \begin{pmatrix} a & -a^2 \\ 1 & -a \end{pmatrix}$ is nilpotent that is $A^2 = 0$.

But if , $A^2 = \rho I$ then this matrix is called involutory , where I is the identity matrix and ρ is scalar .

For example $A = \begin{pmatrix} a & 1-a^2 \\ 1 & -a \end{pmatrix}$ where $\rho = 1$, $A^2 = I$ is involutory

Finally if , $A^3 = \rho A$ then this matrix is called tripotent matrix .

The most important type of submatrices are those whose sets of indices consist of consecutive integers. Such submatrices are called blocks.

For example , in the matrix $\left(\begin{array}{c|cc|c} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \end{array}\right)$

the submatrices $\begin{pmatrix} a_{11} \\ a_{21} \end{pmatrix}$, $\begin{pmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{pmatrix}$, etc are blocks . The above matrix can thus be written as : $A = \begin{pmatrix} P_{11} & P_{12} & P_{13} \\ P_{21} & P_{22} & P_{23} \end{pmatrix}$

where the matrices P_{ik} have the property that all the P_{ik} whose first indices are equal have the same number of rows and all the P_{ik} whose second indices are equal have the same number of columns .

ELEMENTARY MATRICES AND LU FACTORIZATION OF A MATRIX *A*

The process of LU factorization can be described in (n-1) steps as follows

STEP 1 :

Find an elementary matrix E_1 such that $A^{(1)} = E_1 A$ has zeros below the (1,1) entry in the first column , we can find $A^{(1)}$ and E_1 in the form

$$E_1 = \begin{pmatrix} 1 & 0 \; . \; . \; . \; 0 \\ \frac{-a_{21}}{a_{11}} & 1 \; . \; . \; . \; 0 \\ \cdots\cdots & \cdots\cdots \\ \frac{-a_{n1}}{a_{11}} & 0 \; . \; . \; . \; 1 \end{pmatrix} , \quad A^{(1)} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ 0 & a_{22}^{(1)} & \cdots & a_{2n}^{(1)} \\ \cdots & \cdots & \cdots & \cdots \\ 0 & a_{n2}^{(1)} & \cdots & a_{nn}^{(1)} \end{pmatrix}$$

STEP 2 :

Find an elementary matrix E_2 such that $A^{(2)} = E_2 A^{(1)}$ has zeros below the (2,2) entry in the second column ,we can find $A^{(2)}$ and E_2 in the form

$$E_2 = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ 0 & \frac{-a_{32}^{(1)}}{a_{22}^{(1)}} & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & \frac{-a_{n2}^{(1)}}{a_{22}^{(1)}} & \cdots & 1 \end{pmatrix} , \quad A^{(2)} = \begin{pmatrix} a_{11} & a_{12} & & \cdots & a_{1n} \\ 0 & a_{22}^{(1)} & & \cdots & a_{2n}^{(1)} \\ 0 & 0 & a_{33}^{(2)} & \cdots & a_{3n}^{(2)} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & a_{n3}^{(2)} & \cdots & a_{nn}^{(2)} \end{pmatrix}$$

In general at k^{th} step ,an elementary matrix E_k is found such that

$A^{(k)} = E_k A^{(k-1)}$ has zeros below the (k,k) entry in the k^{th} column ,where E_k can be calculated in two steps .

First , an elementary matrix Σ_k of order (n-k+1) is constructed such that

$$\Sigma_k \cdot \begin{pmatrix} a_{kk}^{(k-1)} \\ a_{k+1,k}^{(k-1)} \\ \cdots \\ a_{nk}^{(k-1)} \end{pmatrix} = \begin{pmatrix} a_{kk}^{(k-1)} \\ 0 \\ 0 \\ \cdots \\ 0 \end{pmatrix}$$

and then E_k is defined as $E_k = \begin{pmatrix} I_{k-1} & 0 \\ 0 & \Sigma_k \end{pmatrix}$

At the end of $(n-1)^{th}$ step the matrix $A^{(n-1)}$ is upper triangular

$$A^{(n-1)} = \begin{pmatrix} a_{11} & a_{12} & & \cdots & a_{1n} \\ 0 & a_{22}^{(1)} & & \cdots & a_{2n}^{(1)} \\ 0 & 0 & a_{33}^{(2)} & \cdots & a_{3n}^{(2)} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & a_{nn}^{(n-1)} \end{pmatrix}$$

Now,

$A^{(n-1)} = E_{n-1}A^{(n-2)} = E_{n-1}E_{n-2}A^{(n-3)} = \ldots = \ldots = E_{n-1}\, E_{n-2} \ldots E_2\, E_1 A$. Set $U = A^{(n-1)}$ and $L_1 = E_{n-1}\, E_{n-2} \ldots E_2\, E_1$, then from above we have $U = L_1 A$. Since each E_k is a unit lower triangular matrix, so is the matrix L_1 and therefore , L_1^{-1} exists , set $L_1^{-1} = L$ then we get A=LU .

The elements a_{11} , $a_{22}^{(1)}$, $a_{33}^{(2)}$,…, $a_{nn}^{(n-1)}$ are called pivots and the above process of obtaining LU factorization is known as Gaussian elimination without pivoting .

This method can be applied to an $m \times n$ matrix A (m >1) , in this case we find that the number of steps is min (m-1,n) .

Now , we will show that this factorization is unique as follows :

For if $A=L_1U_1=L_2U_2$ then , since L_1 and L_2 and the matrix A are nonsingular we have det A=det (L_1U_1)=det L_1.detU_1 and det A=det (L_2U_2) = det L_2.detU_2 therefore , U_1 and U_2 are also nonsingular . Hence

$$L_2^{-1}L_1 = U_2U_1^{-1}$$

since $L_2^{-1}L_1$ is a unit lower triangular , $U_2U_1^{-1}$ is upper triangular which are equal only if both are identity and we have $L_1=L_2$, $U_1=U_2$.

This algorithm requires roughly $\frac{n^3}{3}$ flops ,which can be shown as follows :

The first step requires $[(n-1)^2 + (n-1)]$ flops, since we compute (n-1) multipliers and update $(n-1)^2$ entries of A and each multiplier requires one flop and updating each entry also requires one flop.

The second step, similarly , requires $[(n-2)^2 + (n-2)]$ flops . And so on , in general , the k^{th} step requires $[(n-k)^2 + (n-k)]$ flops. Since there are (n-1) steps, then the total flops are :

$$= \sum_{k=1}^{n-1} (n-k)^2 + \sum_{k=1}^{n-1} (n-k)$$

$$= \frac{n(n-1)(2n-1)}{6} + \frac{n(n-1)}{2}$$

$$= \frac{n^3}{3} - \frac{n}{3} \quad .$$

Definition : 1.1.7 [5]

A square matrix is in upper Hessenberg form if all elements below the subdiagonal are zero .

1.2 : MATRIX EXPONENTIAL FUNCTION

The exponential of a square matrix A is defined by the infinite series

$$e^{A} = \sum_{k=0}^{\infty} \frac{A^{k}}{k!} = I + \frac{A}{1!} + \frac{A^{2}}{2!} + \frac{A^{3}}{3!} + \ldots \qquad (1.2.5)$$

which converges for the square matrix A. We can easily sum the right side of (1.2.5) for any diagonal matrix.

Example :

For $A = \begin{pmatrix} 2 & 0 \\ 0 & -0.3 \end{pmatrix}$, we have $e^{A} = \begin{pmatrix} e^{2} & 0 \\ 0 & e^{-0.3} \end{pmatrix}$.

In general, if D is diagonal matrix

$$D = \begin{pmatrix} \lambda_1 & 0 & \ldots & 0 \\ 0 & \lambda_2 & \ldots & 0 \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \ldots & \lambda_n \end{pmatrix} \quad \text{then} \quad e^{D} = \begin{pmatrix} e^{\lambda_1} & 0 & \ldots & 0 \\ 0 & e^{\lambda_2} & \ldots & 0 \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \ldots & e^{\lambda_n} \end{pmatrix}$$

i.e to calculate the exponential of a diagonal matrix, replace each diagonal element by the exponential of that diagonal element.

If a square matrix A is not diagonal, but diagonalizable then there exists a modal matrix M such that $A=MDM^{-1}$ where D is a diagonal matrix. It follows that

$$A^{2} = AA = (MDM^{-1})(MDM^{-1}) = MD(M^{-1}M)DM^{-1} = MD^{2}M^{-1},$$
$$A^{3} = AA^{2} = (MDM^{-1})(MD^{2}M^{-1}) = MD(M^{-1}M)D^{2}M^{-1} = MD^{3}M^{-1}, \ldots$$

and, in general $A^{n} = M\,D^{n}\,M^{-1}$ for any positive integer n. Consequently

$$e^{A} = \sum_{k=0}^{\infty} \frac{A^{k}}{k!} = \sum_{k=0}^{\infty} \frac{MD^{k}M^{-1}}{k!} = M\Big(\sum_{k=0}^{\infty} \frac{D^{k}}{k!}\Big)M^{-1} = Me^{D}M^{-1} \qquad (1.2.6)$$

Example :

For a matrix $A=\begin{pmatrix}1 & 2\\ 4 & 3\end{pmatrix}$ we have , the eigenvalues of A are -1 and 5 with corresponding eigenvectors $[1\ ,\ -1]^T$ and $[1\ ,\ 2]^T$. Here

$$M=\begin{pmatrix}1 & 1\\ -1 & 2\end{pmatrix}\ ,\ M^{-1}=\begin{pmatrix}2/3 & -1/3\\ 1/3 & 1/3\end{pmatrix}\ ,\ \text{and}\ D=\begin{pmatrix}-1 & 0\\ 0 & 5\end{pmatrix}$$

It follows from (1.2.6)

$$e^A=Me^DM^{-1}=\begin{pmatrix}1 & 1\\ -1 & 2\end{pmatrix}\begin{pmatrix}e^{-1} & 0\\ 0 & e^5\end{pmatrix}\begin{pmatrix}2/3 & -1/3\\ 1/3 & 1/3\end{pmatrix}$$

$$=\frac{1}{3}\begin{pmatrix}2e^{-1}+e^5 & -e^{-1}+e^5\\ -2e^{-1}+2e^5 & e^{-1}+2e^5\end{pmatrix}$$

Even if a matrix A is not diagonalizable , it is still similar to a matrix J in Jordan canonical form ,that is, if there exists a general modal matrix M such that $A=MJM^{-1}$ then , we have $e^A=M\,e^J M^{-1}$ (1.2.7) where the exponential of J is obtained from definition (1.1.5) .

Example : For $A=\begin{pmatrix}0 & 4 & 2\\ -3 & 8 & 3\\ 4 & -8 & -2\end{pmatrix}$

we have , a canonical basis for this matrix has one chain of length 2 , $x_1=[0,0,1]^T$ and $x_2=[2\ ,3\ ,-4\ ]^T$ and one chain of length 1 , $y_1=[2\ ,1\ ,\ 0]^T$ each corresponding to the eigenvalue 2 . Setting

$$M=\begin{pmatrix}2 & 2 & 0\\ 1 & 3 & 0\\ 0 & -4 & 1\end{pmatrix}\quad \text{and}\quad J=\begin{pmatrix}2 & 0 & 0\\ 0 & 2 & 1\\ 0 & 0 & 2\end{pmatrix}$$

Since J contains two Jordan blocks , the 1×1 matrix $J_1=[2]$ and the matrix $J_2=\begin{pmatrix}2 & 1\\ 0 & 2\end{pmatrix}$ we have $e^{J_1}=[e^2]$ and $e^{J_2}=\begin{pmatrix}e^2 & e^2\\ 0 & e^2\end{pmatrix}$

then , $e^J = \begin{pmatrix} e^{J_1} & 0 \\ 0 & e^{J_2} \end{pmatrix} = \begin{pmatrix} e^2 & 0 & 0 \\ 0 & e^2 & e^2 \\ 0 & 0 & e^2 \end{pmatrix}$

Since $A = M J M^{-1}$ then

$$e^A = Me^J M^{-1} = \begin{pmatrix} 2 & 2 & 0 \\ 1 & 3 & 0 \\ 0 & -4 & 1 \end{pmatrix} \begin{pmatrix} e^2 & 0 & 0 \\ 0 & e^2 & e^2 \\ 0 & 0 & e^2 \end{pmatrix} \begin{pmatrix} 3/4 & -1/2 & 0 \\ -1/4 & 1/2 & 0 \\ -1 & 2 & 1 \end{pmatrix}$$

$$= e^2 \begin{pmatrix} -1 & 4 & 2 \\ -3 & 7 & 3 \\ 4 & -8 & -3 \end{pmatrix} .$$

1.2.1 : THEORETICAL AND COMPUTATIONAL ASPECTS FOR COMPUTING THE MATRIX EXPONENTIAL e^A

The exponential of a matrix could be computed in many ways . We have methods involving differential equations, the matrix eigenvalues , approximation theory and the matrix characteristic polynomial . Now we will introduce some methods for computing the exponential matrix e^A [7] .

METHOD 1 : Padé approximation

The (m,n) Padé approximation to e^A is defined by

$$R_{mn}(A) = [D_{mn}(A)]^{-1}[N_{mn}(A)]$$

where $N_{mn}(A) = \sum_{i=0}^{m} \frac{(m+n-i)!m!}{(m+n)!i!(m-i)!} A^i$

and $D_{mn}(A) = \sum_{i=0}^{n} \frac{(m+n-i)!n!}{(m+n)!i!(n-i)!} (-A)^i$

Non-singularity of D_{mn} (A) is obtained if the eigenvalues of A are negative

METHOD 2 : Polynomial methods

Since the characteristic polynomial of A is $c(\lambda)=\det(\lambda I-A)=\lambda^n-\sum_{k=0}^{n-1}c_k\lambda^k$

then , by the Cayley-Hamilton theorem we get $c(A)=0$ and hence

$$A^n=c_0I+c_1A+c_2A^2+c_3A^3+\ldots+c_{n-1}A^{n-1}.$$

Also ,we can find that any power of A can be expressed in terms of

$$I, A, A^2, A^3, \ldots, A^{n-1} : \qquad A^k=\sum_{i=0}^{n-1}\beta_{ki}A^i$$

for this we find that e^{tA} is a polynomial in A with analytic coefficients in t:

$$e^{tA}=\sum_{k=0}^{\infty}\frac{t^kA^k}{k!}=\sum_{k=0}^{\infty}\frac{t^k}{k!}\Big[\sum_{i=0}^{n-1}\beta_{ki}A^i\Big]$$

$$=\sum_{i=0}^{n-1}\Big[\sum_{k=0}^{\infty}\beta_{ki}\frac{t^k}{k!}\Big]A^i=\sum_{i=0}^{n-1}\alpha_i(t)A^i\ .$$

where $\sum_{k=0}^{\infty}\beta_{ki}\dfrac{t^k}{k!}=\alpha_i(t)$

METHOD 3 : Lagrange interpolation

$$e^{tA}=\sum_{i=0}^{n-1}e^{\lambda_i t}\prod_{\substack{k=1\\ k\neq i}}^{n}\frac{(A-\lambda_k I)}{(\lambda_i-\lambda_k)}$$

where $\lambda_1, \lambda_2, \lambda_3, \ldots, \lambda_n$ are the eigenvalues of A .

METHOD 4 : Newton interpolation

$$e^{tA}=e^{\lambda_1 t}I+\sum_{i=2}^{n}[\lambda_1,\ldots,\lambda_i]\prod_{k=1}^{i-1}(A-\lambda_k I)$$

where , the divided differences $[\lambda_1, \ldots, \lambda_i]$ depend on t and are defined recursively by

$$[\lambda_1,\lambda_2]=(e^{\lambda_1 t}-e^{\lambda_2 t})/(\lambda_1-\lambda_2),$$

$$[\lambda_1,\ldots,\lambda_{k+1}]=\frac{[\lambda_1,\ldots,\lambda_k]-[\lambda_2,\ldots,\lambda_{k+1}]}{\lambda_1-\lambda_{k+1}} \qquad (k\geq 2)$$

METHOD 5 : Inverse Laplace transformation

First , consider the state differential equation

$$\dot{x}(t)=A\,x(t)$$

which have the solution

$$x(t)=e^{At}x(0) \qquad (1.2.8)$$

where x (0) is the initial solution at t =0 .

If we take the Laplace transform of the differential equation we obtain ,

$s\,x(s)-x(0)=A\,x(s)$ i.e $(sI-A)x(s)=x(0)$

we see that provided s is such that (sI-A) is invertible , we can solve

the last equation for x (s) as $x(s)=(sI-A)^{-1}x(0)$ (1.2.9)

Finally, taking the inverse Laplace transform of (1.2.9) yields

$$x(t)=£^{-1}[(sI-A)^{-1}]\,x(0)$$

and we can see , by comparing this equation with (1.2.8) we have

$$e^{At}=£^{-1}[(sI-A)^{-1}]$$

Now, since $(sI-A)^{-1}=\sum_{k=0}^{n-1}\frac{s^{n-k-1}}{c(s)}A_k$

such that $c(s)=\det(sI-A)=s^n-\sum_{k=0}^{n-1}c_k s^k$ and for $k=1,2,\ldots,n$

we have $c_{n-k}=-\mathrm{trace}(A_{k-1}A)/k$, $A_k=A_{k-1}A-c_{n-k}I$, $(A_0=I)$

from the above recursion relations we can evaluate e^{tA} as :

$$e^{tA}=\sum_{k=0}^{n-1}£^{-1}\left[\frac{s^{n-k-1}}{c(s)}\right]A_k$$

The inverse transforms $\pounds^{-1}[\dfrac{s^{n-k-1}}{c(s)}]$ can be expressed as a power series in t .

METHOD 6: Companion matrix

We now discuss technique which involve the computation of e^C where C is a companion matrix defined by definition (1.1.3) .

Since the companion matrix C has the characteristic polynomial

$$c(z) = z^n - \sum_{k=0}^{n-1} c_k z^k$$

for this we can apply method (2) to evaluate e^C . If $A = YCY^{-1}$, then from the series definition of the matrix exponential it is easy to verify that $e^A = Y e^C Y^{-1}$.

METHOD 7 : Schur method

The Schur decomposition of any matrix A is $A = DSD^T$ with orthogonal matrix D and triangular S ,(exists if A has real eigenvalues). But if A has complex eigenvalues ,we will take D and S as complex (and replace D^T with D^*). Now , we can find that $e^{tA} = De^{tS}D^T$. Also , if S is upper triangular with diagonal elements $\lambda_1,\ldots,\lambda_n$ then the exponential matrix e^{tS} , is also upper triangular with diagonal elements $e^{\lambda_1 t},\ldots,e^{\lambda_n t}$.

METHOD 8 : Splitting method

We can approximate e^A by splitting A into B+C and then using

$$e^{tB}e^{tC} = e^{t(B+C)} \qquad \Leftrightarrow \qquad BC = CB$$

This approach to computing e^A is of potential interest when the exponential of B and C can be accurately and efficiently computed .

1.2.2: EXPLICIT COMPUTATION TO THE MATRIX EXPONENTIAL

Lemma :1.2.8 [3]

Let $A=\begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \in C^{2\times 2}$

1- If $a=d$ then $e^A = e^a\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}$

2- If $a \neq d$ then $e^A = \begin{pmatrix} e^a & b(e^a-e^d)/(a-d) \\ 0 & e^d \end{pmatrix}$.

Proof :

1- If $a = d$, we can see that

$$A=\begin{pmatrix} a & b \\ 0 & a \end{pmatrix},\ A^2=\begin{pmatrix} a^2 & 2ab \\ 0 & a^2 \end{pmatrix},\ A^3=\begin{pmatrix} a^3 & 3a^2b \\ 0 & a^3 \end{pmatrix},\ \ldots$$

Now from the power series definition of e^A (1.2.5) and by substituting from $I, A, A^2, A^3, \ldots$ into (1.2 .5) we get

$$e^A=\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}+\begin{pmatrix} a & b \\ 0 & a \end{pmatrix}+\frac{1}{2!}\begin{pmatrix} a^2 & 2ab \\ 0 & a^2 \end{pmatrix}+\frac{1}{3!}\begin{pmatrix} a^3 & 3a^2b \\ 0 & a^3 \end{pmatrix}+\ldots=e^a\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}.$$

2- If $a \neq d$ we can see that

$$A=\begin{pmatrix} a & b \\ 0 & d \end{pmatrix}, A^2=\begin{pmatrix} a^2 & ab+bd \\ 0 & d^2 \end{pmatrix}, A^3=\begin{pmatrix} a^3 & a^2b+abd+bd^2 \\ 0 & d^3 \end{pmatrix},\ldots$$

from the power series definition of e^A and substituting from I, A, A^2, $A^3, \ldots$ into (1.2 .5) we get

$$e^A=\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}+\begin{pmatrix} a & b \\ 0 & d \end{pmatrix}+\frac{1}{2!}\begin{pmatrix} a^2 & ab+bd \\ 0 & d^2 \end{pmatrix}+\frac{1}{3!}\begin{pmatrix} a^3 & a^2b+bd^2+abd \\ 0 & d^3 \end{pmatrix}+\ldots$$

$$=\begin{pmatrix} e^a & b(e^a-e^d)/(a-d) \\ 0 & e^d \end{pmatrix}.$$

Lemma :1.2.9 [3]

If $A^2 = \rho I$ for any matrix $A \in C^{n\times n}$, where $\rho \in C$

1- If $\rho = 0$ then $e^A = I + A$.

2- If $\rho \neq 0$ then $e^A = (\cosh\sqrt{\rho})I + (\frac{\sinh\sqrt{\rho}}{\sqrt{\rho}})A$.

Proof :

1- Directly from $e^A = \sum_{k=0}^{\infty} \frac{A^k}{k!}$ for $A^2 = \rho I = 0$ then $e^A = I + A$.

2- Since $A^2 = \rho I$, $A^{2k} = \rho^k I$, $A^{2k+1} = \rho^k A$ for $k \geq 0$ then

$$e^A = (I + \frac{A^2}{2!} + \frac{A^4}{4!} + ...) + (A + \frac{A^3}{3!} + \frac{A^5}{5!} + ...) = (1 + \frac{\rho}{2!} + \frac{\rho^2}{4!} + ...)I$$

$$+ (1 + \frac{\rho}{3!} + \frac{\rho^2}{5!} + ...)A = (\cosh\sqrt{\rho})I + (\frac{\sinh\sqrt{\rho}}{\sqrt{\rho}})A .$$

where, if $A^2=0$ then the matrix A is called nilpotent and if $A^2 = \rho I$ is called involutory .

Theorem : 1.2.10 [3]

Let $A \in C^{n\times n}$ and suppose that $A^3 = \rho A$, where $\rho \in C$

1- If $\rho = 0$ then $e^A = I + A + \frac{A^2}{2}$.

2- If $\rho \neq 0$ then $e^A = I + (\frac{\sinh\sqrt{\rho}}{\sqrt{\rho}})A + (\frac{\cosh\sqrt{\rho} - 1}{\rho})A^2$.

Proof :

1- Directly from $e^A = \sum_{k=0}^{\infty} \frac{A^k}{k!}$ with $A^3 = 0$, then $e^A = I + A + \frac{A^2}{2}$.

2- Since $A^3 = \rho A$, $A^{2k+2} = \rho^k A^2$, $A^{2k+1} = \rho^k A$ for $k \geq 0$. Then

$$e^{A}=(I+\frac{A^{2}}{2!}+\frac{A^{4}}{4!}+\ldots)+(A+\frac{A^{3}}{3!}+\frac{A^{5}}{5!}+\ldots)=I+(1+\frac{\rho}{3!}+\frac{\rho^{2}}{5!}+\ldots)A$$
$$+(\frac{1}{2!}+\frac{\rho}{4!}+\ldots)A^{2}\quad =I+(\frac{\sinh\sqrt{\rho}}{\sqrt{\rho}})A+(\frac{\cosh\sqrt{\rho}-1}{\rho})A^{2}.$$

where the matrix A which satisfy that $A^{3}=\rho A$ is called tripotent matrix.

1.3:STATE TRANSITION MATRIX AND TRANSIENT RESPONSE VECTORS COMPUTATIONS

For a linear time-invariant system described by the vector matrix

equation $\dot{x}=Ax+u(t)\,,\qquad x(0)=x_{0}\qquad (1.3.10)$

where x (t) is the n-vector of states, u (t) is an n-vector of controls and A is the constant coefficients matrix. The solution of equation (1.3.10) can be

written in the form $x(t)=e^{At}x_{0}+\int_{0}^{t}e^{A(t-\tau)}u(\tau)d\tau$

In the above equation e^{At} , which is a function of time , is the state transition matrix . For a force-free system the response vector given by $x(t)=e^{At}x_{0}$ where the square matrix e^{At} is referred to as " The matrix exponential of At" where it can be expressed as an infinite series

$$e^{At}=I+At+\frac{A^{2}t^{2}}{2!}+\frac{A^{3}t^{3}}{3!}+\ldots=\sum_{r=0}^{\infty}\frac{A^{r}t^{r}}{r!}\qquad (1.3.11)$$

Now, we can show that the transition matrix given by (1.3.11) solves the

state differential equation $\dot{x}(t)=Ax(t)\qquad (1.3.12)$

As follows , at first we differentiate the forgoing series expansion for the matrix exponential of At (1.3.11) to obtain

$$\frac{d(e^{At})}{dt} = \frac{d}{dt}\left(\sum_{r=0}^{\infty} \frac{A^r t^r}{r!}\right) = \sum_{r=0}^{\infty} \frac{d}{dt}\left(\frac{A^r t^r}{r!}\right) = \sum_{r=0}^{\infty} \frac{rA^r t^{r-1}}{r!}$$

$$= 0 + \sum_{r=1}^{\infty} \frac{AA^{r-1} t^{r-1}}{(r-1)!} = A\left(\sum_{r=1}^{\infty} \frac{A^{r-1} t^{r-1}}{(r-1)!}\right) = A\left(\sum_{k=0}^{\infty} \frac{A^k t^k}{k!}\right) = Ae^{At} \qquad \text{(I)}$$

Then using this relation to differentiate the assumed solution

$$x(t) = e^{At} x_0 \qquad \text{(II)}$$

yields $\dot{x}(t) = Ae^{At} x_0 = A x(t)$.

therefore , from I , II we can see $x(t) = e^{At} x_0$ solves the state differential equation (1.3.12) .

Now, for a chosen interval of time τ, which need not be very small , $e^{A\tau}$ can be approximated by the first few terms retaining any desired accuracy . If $e^{A\tau}$ is the matrix thus evaluated the response vector will be given by

$$x(k\tau) = e^{k\tau A} x_0 \quad , \quad k = 1, 2, 3, \ldots \qquad (1.3.13)$$

for a force-free system .

1.3.1 : CAYLEY-HAMILTON METHOD FOR COMPUTING THE TRANSIENT RESPONSE VECTORS

In this section , I will introduce a numerical technique that has been proposed to evaluate the transient response vectors based on the characteristic polynomial of A making use of the Cayley-Hamilton theorem as in [13] to approximate e^{tA}. This method is a modification of the method of Liou [18] . We will introduce this method in a series of steps as follows :

STEP 1 :

For a given square matrix A of a system ,compute $A^2, A^3, \ldots, A^{n-1}$

STEP 2 : Formulate the characteristic equation

$$(-1)^n|A-\lambda I| = \lambda^n + p_1\lambda^{n-1} + \ldots + p_n = 0 \qquad (1.3.14)$$

where p_i (i = 1(1)n) can be computed by the expansion of the determinant .

STEP 3 : From Cayley-Hamilton theorem we have

$$A^n = -p_1A^{n-1} - p_2A^{n-2} - \ldots - p_nI \qquad (1.3.15)$$

successive multiplication of (1.3.15) by A gives rise to the following equations expressing A^{n+q} in terms $A, A^2, A^3, \ldots, A^{n-1}$ as

$$A^{n+q} = \alpha_{0q}I + \alpha_{1q}A + \alpha_{2q}A^2 + \ldots + \alpha_{n-1,q}A^{n-1}, \qquad q = 0,1,2,\ldots \quad (1.3.16)$$

where $\alpha_{00} = -p_n$, $\alpha_{10} = -p_{n-1}$, . . ., $\alpha_{n-1,0} = -p_1$, since α_{0q}, α_{1q}, α_{2q}, . . ., $\alpha_{n-1,q}$ are obtained from the following recurrence relationships

$$\alpha_{0q} = -p_n\,\alpha_{0,q-1}$$

$$\alpha_{1q} = \alpha_{0,q-1} - p_{n-1}\,\alpha_{1,q-1}$$

$$\alpha_{2q} = \alpha_{1,q-1} - p_{n-2}\,\alpha_{2,q-1}$$

....

$$\alpha_{n-1,q} = \alpha_{n-2,q-1} - p_1\,\alpha_{n-1,q-1}$$

STEP 4: For a given choice of τ then $e^{\tau A}$ can be written as

$$e^{\tau A} = I + \tau A + \frac{\tau^2}{2!}A^2 + \frac{\tau^3}{3!}A^3 + \ldots + \frac{\tau^n}{n!}A^n + \ldots$$

$$= I + \tau A + \frac{\tau^2}{2!}A^2 + \ldots + \frac{\tau^n}{n!}(\alpha_{00}I + \alpha_{10}A + \ldots + \alpha_{n-1,0}A^{n-1})$$

$$+ \frac{\tau^{n+1}}{(n+1)!}(\alpha_{01}I + \alpha_{11}A + \alpha_{21}A^2 + \ldots + \alpha_{n-1,1}A^{n-1}) + \ldots$$

$$= I(1 + \frac{\alpha_{00}\tau^n}{n!} + \frac{\alpha_{01}\tau^{n+1}}{(n+1)!} + \ldots) + A(\tau + \frac{\alpha_{10}\tau^n}{n!} + \frac{\alpha_{11}\tau^{n+1}}{(n+1)!} + \ldots) + \ldots$$

$$+ A^{n-1}(\frac{\tau^{n-1}}{(n-1)!} + \frac{\alpha_{n-1,1}\tau^{n+1}}{(n+1)!} + \frac{\alpha_{n-1,0}\tau^n}{n!} + \ldots)$$

Now, since we can compute the matrices $A^2, A^3, \ldots, A^{n-1}$ and the α's from step 3 then $e^{A\tau}$ can be calculates to any desired accuracy without actual series summation of power of A higher than (n-1) .

STEP 5 :

Suppose that $e^{A\tau} = M(\tau)$ as evaluated from step 4 and compute $M^2, M^3, \ldots, M^{n-1}$ and evaluate the characteristic equation of M

$$(-1)^n |M - \lambda I| = \lambda^n + m_1 \lambda^{n-1} + \ldots + m_n = 0 \qquad (1.3.17)$$

where m_i (i= 1, 2, . . . , n) can be found by expansion of the determinant

STEP 6 :

From (1.3.13) we have $\quad x(k\tau) = M^k(\tau)x(0) \qquad (1.3.18)$

we can compute the matrices $M, M^2, \ldots, M^{n-1}$, and M^k for $(k \geq n)$ if we apply the Cayley-Hamilton theorem for $M(\tau)$ as follows , from (1.3.17) we have $\quad M^n = -m_1 M^{n-1} - m_2 M^{n-2} - \ldots - m_n I$

and therefore , $\quad M^{n+p} = \beta_{0p} I + \beta_{1p} M + \ldots + \beta_{n-1,p} M^{n-1}$

where $\beta_{00} = -m_n, \beta_{10} = -m_{n-1}, \ldots, \beta_{n-1,0} = -m_1$ for $p = 0, 1, \ldots$

and $\beta_{0p}, \beta_{1p}, \ldots, \beta_{n-1,p}$ can be evaluates from the previous recurrence relationship with α's changed to β's and A changed to M .

STEP 7 :

For $k \geq n$ $\quad x(k\tau) = M^{n+p} x(0)$ where $p = k - n \geq 0$ therefore

$$x(k\tau) = (\beta_{0p} I + \beta_{1p} M + \ldots + \beta_{n-1,p} M^{n-1}) x(0)$$

$$= \beta_{0p} x(0) + \beta_{1p} x(\tau) + \beta_{2p} x(2\tau) + \ldots + \beta_{n-1,p} x(\tau(n-1)) \qquad (1.3.19)$$

the equation (1.3.19) expresses the vectors $x_1, x_2, \ldots, x_n$ at any instance of time $t = k\tau$ as a linear combination of n vectors

$$x(0), x(\tau), \ldots, x(\tau(n-1) \quad \text{at} \quad t = 0, \tau, \ldots, (n-1)\tau \;.$$

Numerical example [13]

We consider the system described by $\dot{x} = A\,\mathrm{x}$ where

$\mathrm{x}(0) = [2 \quad -2.5 \quad 3.75]^T$ and $$A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -0.75 & -2.75 & -3 \end{pmatrix}.$$

The exact solution is $$\mathrm{x(t)} = \begin{pmatrix} 2e^{-1.5t} + e^{-0.5t} - e^{-t} \\ -0.5e^{-0.5t} + e^{-t} - 3e^{-1.5t} \\ 0.25e^{-0.5t} - e^{-t} + 4.5e^{-1.5t} \end{pmatrix}.$$

We compute the transient response vectors as follows :

STEP 1 :

$$A^2 = \begin{pmatrix} 0 & 0 & 1 \\ -0.75 & -2.75 & -3 \\ 2.25 & 7.50 & 6.25 \end{pmatrix}$$

STEP 2 :

The characteristic equation of A is $\lambda^3 + 3\lambda^2 + 2.75\lambda + 0.75 = 0$

STEP 3 :

$$A^3 = -0.75I - 2.75A - 3A^2 \,,\, \mathrm{A}^4 = 2.25I + 7.5A + 6.25A^2$$
$$A^5 = 4.6875I - 14.9375A + 11.25A^2$$

STEP 4 :

For $\tau = 0.1$

$$\mathrm{M} = e^{0.1\mathrm{A}} = \begin{pmatrix} 0.999884 & 0.0995717 & 0.00452513 \\ -0.00339385 & 0.987441 & 0.0859963 \\ -0.0644972 & -0.239884 & 0.729451 \end{pmatrix}$$

STEP 5 : $$M^2 = \begin{pmatrix} 0.999138 & 0.196795 & 0.016388 \\ -0.012291 & 0.954072 & 0.147631 \\ -0.1123515 & -0.4182765 & 0.561178 \end{pmatrix}$$

and the characteristic equation of M is

$$\lambda^3 - 2.716775\lambda^2 + 2.458239\lambda - 0.7408182 = 0$$

STEP 6 :

We can calculate the coefficients (for n=3) from the recurrence relationships with the starting values :

$$\beta_{00} = 0.7408182 \ , \beta_{10} = -2.458239, \beta_{20} = 2.716775 \ .$$

STEP 7 :

The solutions $x_i(t)$ are computed at intervals of 0.1 and are given in the following table :

Comparison of the results between Ganapathy and Rao method with the exact solution .

$t = k\tau$	$x_{i_1}(k\tau)$	Exact solution
0.1	1.76781	1.76781
0.2	1.56774	1.56774
0.3	1.39515	1.39515
0.4	1.24603	1.24604
0.5	1.11700	1.11700
0.6	1.00514	1.00515
0.7	0.907977	0.907979
0.8	0.823377	0.823379
0.9	0.749536	0.749538
1	0.684908	0.684912

In the above table ,we computed the first component for each vector $x(k\tau)$ for comparison with the corresponding component in the exact solution .