# Chapter 1

# Preliminaries

## 1.1 Introduction

In this chapter, we give some basic definitions and known results on oscillatory behaviour of solutions of nonlinear second order difference equations.

We state and prove also some known results in stability of the zero solution of nonlinear systems of difference equations.

In 1985 Szmanda [28] discussed the oscillation of solutions of the second order nonlinear difference equations of the form

$$\Delta(r_n \Delta x_n) + q_n f(n, x_n) = g(n, x_n), \quad n \in N(n_o), \tag{1.1.1}$$

and the special case of Eq.(1.1.1)

$$\Delta(r_n \Delta x_n) + f(n, x_n) = g(n, x_n), \quad n \in N(n_o), \tag{1.1.2}$$

where $f,\ g\ :\ N(n_o) \times \mathbb{R} \to \mathbb{R}$ and $\{r_n\}$, $\{q_n\}$ are the real sequences. Further, $r_n > 0$ for all $n \in N(n_o)$, and $R_{n_o,n} \to \infty$ as $n \to \infty$, where

$$R_{n_o,n} = \sum_{l=n_o}^{n-1} \frac{1}{r_l}, \quad \text{for all } n \in N(n_o).$$

For the difference equation (1.1.1) each result depends on some of the following conditions.

($c_1$) $q_n > 0$ for all $n \in N(n_o)$,

($c_2$) there exists a constant $M_1$ such that

$$f(n, x) \geq M_1,$$

($c_3$) there exists a constant $M_2$ such that

$$f(n, x) \leq M_2,$$

($c_4$) there exists a real sequence $\{\bar{b_n}\}$ such that

$$g(n, x) \geq \bar{b_n},$$

($c_5$) there exists a real sequence $\{b_n\}$ such that

$$g(n, x) \leq b_n,$$

($c_6$) $f(n, x)$ is bounded from above if $x$ is bounded,

($c_7$) $f(n, x)$ is bounded from below if $x$ is bounded,

($c_8$) $xf(n, x) \geq 0$,

($c_9$) $xf(n, x) \leq 0$,

($c_{10}$) there are real sequences $\{\bar{p_n}\}$ and $\{p_n\}$ such that

$$\bar{p_n} \leq f(n, x) \leq p_n.$$

The second order nonlinear difference equation

$$\Delta^2 x_n + q_{n+1} f(x_{n+1}) = 0, \quad n \geq 0, \tag{1.1.3}$$

where $f \in C(\mathbb{R}, \mathbb{R})$, $f$ is nondecreasing, $x\,f(x) > 0$ as $x \neq 0$, and $\{q_{n+1}\}$ is real sequence, has been studied in 1994 by Thandapani, Györi, and Lalli [31], and in 1996 by Zhang and Chen [37].

In [1], [10], [18], [23], [31], and [33] the authors discussed the oscillation and nonoscillation of solutions of second order nonlinear difference equations

$$\Delta(r_n \Delta x_n) + q_n f(x_n) = 0 \quad n \in N(n_o), \tag{1.1.4}$$

and

$$\Delta(r_n \Delta x_n) + f(n, x_n) = 0 \quad n \in N(n_o). \tag{1.1.5}$$

## 1.2 Oscillation of nonlinear difference equations

In this section, we state and prove some known results on second order nonlinear difference equations of the types (1.1.1), (1.1.3), and (1.1.4).

The following definitions will be needed.

**Definition 1.2.1.** [1]: The sequence $\{x_n\}$ of real numbers is said to be eventually positive (negative), if there exists $m > 0$, such that

$$x_n > 0 \quad (x_n < 0), \quad \text{for all } n \geq m.$$

**Definition 1.2.2.** [2]: The sequence $\{x_n\}$ is said to be strictly oscillatory around 0, or strictly oscillatory, or simply oscillatory if for every $m > 0$, there exists an $n \geq m$ such that

$$x_n \, x_{n+1} \leq 0.$$

Otherwise, the sequence $\{x_n\}$ is said to be nonoscillatory. Equivalently, $\{x_n\}$ is nonoscillatory if it is eventually positive or eventually negative.

**Definition 1.2.3.** [10]: The function $f : N(n_o) \times \mathbb{R} \to \mathbb{R}$ is called

**(1)** Superlinear if for each $n$

$$\frac{f(n, x_1)}{x_1} \geq \frac{f(n, x_2)}{x_2},$$

for $x_1 \geq x_2 > 0$ or $x_1 \leq x_2 < 0$.