Theorem 2.3.1. [26]: *(*Schauder's fixed point theorem*)*
If K is a closed, bounded, and convex subset of a Banach space and the mapping $T : K \to K$ is completely continuous, then T has a fixed point in K.

We first give the following two Lemmas will be needed for later results in this section.

Lemma 2.3.1. *Suppose that φ is a closed, bounded, and convex subset of a Banach space X such that*

$$\varphi = \left\{ x \in X |\, a \leq x_n \leq a + \sum_{k=n_o}^{n-1} \frac{1}{r_k \psi(a)} \sum_{i=k}^{\infty} q_i\, f(i,a),\ n \geq n_o \right\},$$

and $T : \varphi \to \varphi$, be such that

$$(Tx)_n = a + \sum_{k=n_o}^{n-1} \frac{1}{r_k \psi(x_k)} \sum_{i=k}^{\infty} q_i f(i, x_i), \tag{2.3.1}$$

where $a > 0$, $\{q_n\}$ is positive sequence and $f(n,x)$ is nonincreasing in $x \in (0,\infty)$, for $n \geq n_o$.

Then T satisfies the Schauder's fixed point Theorem 2.3.1.

Proof. We show that T satisfies the following

(1) T maps φ into φ.
Since $f(n,x)$ is nonincreasing in $x \in (0,\infty)$, then by the assumptions (A_7) and (A_8), it follows that $\psi(x)$ is nondecreasing positive function.

Indeed, if $x \in \varphi$, then

$$(Tx)_n \geq a, \quad \text{for all} \quad n \in N(n_o).$$

But since $x_n \geq a$ for all $n \in N(n_o)$, we get

$$\psi(x_n) \geq \psi(a),$$

i.e.

$$\frac{1}{r_n\psi(x_n)} \le \frac{1}{r_n\psi(a)},$$

and

$$q_nf(n,x_n) \le q_nf(n,a).$$

Then

$$\sum_{k=n_o}^{n-1}\frac{1}{r_k\psi(x_k)}\sum_{i=k}^{\infty}q_if(i,x_i) \le \sum_{k=n_o}^{n-1}\frac{1}{r_k\psi(a)}\sum_{i=k}^{\infty}q_if(i,a),$$

for all $n \in N(n_o)$ and $a > 0$.

Hence, we obtain

$$(Tx)_n \le a + \sum_{k=n_o}^{n-1}\frac{1}{r_k\psi(a)}\sum_{i=k}^{\infty}q_if(i,a).$$

(2) T is continuous.

In fact , if $\epsilon > 0$, then we may choose $N \ge n_o$ so large such that

$$\sum_{k=N}^{\infty}q_kf(k,a) < \epsilon, \qquad \text{for all} \quad n \ge N.$$

Also if $\{x_n^v\}_{v=1}^{\infty}$ is a sequence of elements of φ such that $x^v \to x$ as $v \to \infty$. Hence since φ is closed, $x \in \varphi$, for all large v, then it follows that

$$\begin{aligned}
&|(Tx^v)_n - (Tx)_n| \\
&\quad = \left|\sum_{k=n_o}^{n-1}\frac{1}{r_k\psi(x_k^v)}\sum_{i=k}^{\infty}q_if(i,x_i^v) - \sum_{k=n_o}^{n-1}\frac{1}{r_k\psi(x_k)}\sum_{i=k}^{\infty}q_if(i,x_i)\right| \\
&\quad \le \left|\sum_{k=n_o}^{n-1}\frac{1}{r_k\psi(x_k^v)}\sum_{i=k}^{\infty}q_if(i,x_i^v) - \sum_{k=n_o}^{n-1}\frac{1}{r_k\psi(x_k)}\sum_{i=k}^{\infty}q_if(i,x_i^v)\right| \\
&\qquad + \left|\sum_{k=n_o}^{n-1}\frac{1}{r_k\psi(x_k)}\sum_{i=k}^{\infty}q_if(i,x_i^v) - \sum_{k=n_o}^{n-1}\frac{1}{r_k\psi(x_k)}\sum_{i=k}^{\infty}q_if(i,x_i)\right|,
\end{aligned}$$

i.e.

$$|(Tx^v)_n - (Tx)_n| \leq \sum_{k=n_o}^{n-1} \frac{1}{r_k} \left| \frac{\psi(x_k^v) - \psi(x_k)}{\psi(x_k^v)\psi(x_k)} \right| \sum_{i=k}^{\infty} q_i f(i, x_i^v)$$
$$+ \sum_{k=n_o}^{n-1} \frac{1}{r_k \psi(x_k)} \sum_{i=k}^{\infty} q_i \left| f(i, x_i^v) - f(i, x_i) \right|$$
$$\leq \sum_{k=n_o}^{n-1} \frac{\lambda}{r_k} \left| \psi(x_k^v) - \psi(x_k) \right| \sum_{i=k}^{\infty} q_i f(i, x_i^v)$$
$$+ R_{n_o,n}(a) \sum_{i=n_o}^{\infty} q_i \left| f(i, x_i^v) - f(i, x_i) \right|,$$

where

$$\lambda \geq \frac{1}{[\psi(a)]^2} \geq \frac{1}{\psi(x_k^v)\psi(x_k)}, \quad \text{and } R_{n_o,n}(a) = \sum_{k=n_o}^{n-1} \frac{1}{r_k \psi(a)}.$$

Then

$$|(Tx^v)_n - (Tx)_n| \leq \lambda \sum_{k=n_o}^{n-1} \frac{1}{r_k} \left| \psi(x_k^v) - \psi(x_k) \right| \sum_{i=k}^{\infty} q_i f(i, x_i^v)$$
$$+ R_{n_o,n}(a) \sum_{i=n_o}^{N} q_i \left| f(i, x_i^v) - f(i, x_i) \right|$$
$$+ R_{n_o,n}(a) \sum_{i=N}^{\infty} q_i \left| f(i, x_i^v) - f(i, x_i) \right|.$$

Hence, since $\psi(x)$ and f are continuous functions and $x^v \to x$ as $v \to \infty$, then following Li [23], we conclude that

$$|(Tx^v)_n - (Tx)_n| \to 0 \quad \text{as} \quad v \to \infty.$$

Consequently T is continuous.

(3) $T\varphi$ is uniformly Cauchy.

Let $x \in \varphi$ and $m, n \geq n_o$.