CHAPTER 0

INTRODUCTION

In 1965 Zadeh [84], it is introduced the idea of a fuzzy set as an extension of classical set theory. This chapter is considerd as a background for the material inclouded in this thesis. It contains, fuzzy subsets, fuzzy supratopology, fuzzy (bi) topology, fuzzy (quasi-) proximity and dimension theory of classical bitopological spaces.

In classical set theory, an element either belong to a set or not belong to this set. If X is a set of objects, membership in a classical subset A of X is often viewed as a characteristic function 1_A from X to the set {0,1} which is called a valuation set such that :

$$1_A(x) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{if } x \notin A \end{cases}$$

If the valuation set is the closed unit interval I = [0,1], A is called a fuzzy set. In other words, a fuzzy set can be regarded as a function from X into I. Fuzzy sets on X will be denoted by Greek litters as μ, ρ, η etc.

An ordinary subset A of X can be regarded as a fuzzy set from X into the set {0,1} and in this case it is called crisp subset of X. Crisp subsets of X will be denoted by the capital letters A,B,C etc.

If $\mu : X \longrightarrow I$ is a fuzzy set, then $\mu(x)$ is the value of μ at $x \in X$ and it is called the degree of the membership of x in μ.

0.1. FUZZY SETS.

Definition 0.1.1 [84]. Let X be a set. A fuzzy subset of X is a function $\mu : X \longrightarrow I$. The fuzzy subsets $\emptyset$ and X are the constant functions taking whole of X to 0 and 1, respectively. We denote by I^X, the class of all fuzzy subsets in X.

Definition 0.1.2 [84]. Let μ, $\rho \in I^X$. Then

(a) $\mu \subseteq \rho$ iff $\mu(x) \leq \rho(x)$, for all $x \in X$.

(b) $\mu' = X \backslash \mu$ iff $\mu'(x) = 1-\mu(x)$, for each $x \in X$, μ' is the complement of μ.

(c) For a collection $\{\mu_j : j \in J\}$ of fuzzy subsets of X, the union $\bigcup_{j \in J} \mu$ and the intersection $\bigcap_{j \in J} \mu$ are fuzzy subsets of X and defined as follows : $(\bigcup_{j \in J} \mu_j)(x) = \sup_{x \in X}\{\mu_j(x)\}_{j \in J}$ and $(\bigcap_{j \in J} \mu_j)(x) = \inf_{x \in X}\{\mu_j(x)\}_{j \in J}$ for each $x \in X$.

Definition 0.1.3 [20,52,68]. Let μ, $\rho \in I^X$. Then

(a) $\mu_{S\alpha} = \{x \in X : \mu(x) > \alpha\}$ is called the strong α-cut of μ, where $\alpha \in [0,1)$.

The strong 0-cut of μ is called the support of μ and is denoted by supp(μ).

(b) $\mu_{W\alpha} = \{x \in X : \mu(x) \geq \alpha\}$ is called the weak α-cut of μ, $\alpha \in (0,1]$.

(c) $hgt(\mu) = Sup_{x \in X}\ \mu(x)$ is called the height of μ.

(d) A fuzzy singleton x_t (or a fuzzy point) in X is a fuzzy set defined by :

$$x_t(y) = \begin{cases} t & \text{if } y = x \\ 0 & \text{if } y \neq x \end{cases}$$

where the point $x \in X$ and the value $t \in (0,1]$ are respectively the support and the value of the singleton x_t.

(e) A fuzzy singleton $x_t \in \mu$ iff $t \leq \mu(x)$.

We denote by S(X), the set of all fuzzy singletons or fuzzy points in X.

Proposition 0.1.4 [68]. Let $\mu, \rho, \eta \in I^X$ and $\{\mu_j : j \in J\} \subseteq I^X$. Then

(a) $\mu = \bigcup_{x_t \in \mu} (x_t) = \bigcup_{x_t q \mu} (x_{1-t})$.

(b) $x_t \in \bigcap_{j \in J} \mu_j$ iff $x_t \in \mu_j$ for all $j \in J$.

(c) $x_t \in \bigcup_{j=J}^{n} \mu_j$ iff there exists $j \in \{1,2,\dots,n\}$ such that $x_t \in \mu_j$.

(d) If there exists $j \in J$ such that $x_t \in \mu_j$, then $x_t \in \bigcup_{j \in J} \mu_j$.

Definition 0.1.5 [69]. A fuzzy set μ is said to be quasi-coincident with a fuzzy set ρ, in symbol $\mu \; q \; \rho$, iff there exists $x \in X$ such that $\mu(x) + \rho(x) > 1$. In particular, $x_t \; q \; \mu$ iff $t + \mu(x) > 1$. If μ is not quasi-coincident with ρ then we write $\mu \; \bar{q} \; \rho$.

Proposition 0.1.6 [20,69]. Let $\mu, \rho, \eta \in I^X$ and $\{\mu_j : j \in J\} \subseteq I^X$. Then

(a) $\mu \; q \; \rho$ implies $\mu \cap \rho \neq \emptyset$.

(b) $\mu \; \bar{q} \; \rho$ iff $\mu \leq \rho'$.

(c) $\mu \; q \; \rho$ iff there exists $x_t \in S(X)$ such that $x_t \in \mu$ and $x_t \; q \; \rho$.

(d) For all $x, y \in X$, $x \neq y$ implies $x_t \; \bar{q} \; y_r$. If $x_t \; \bar{q} \; y_r$, then $x \neq y$ or $x=y$ and $t + r \leq 1$.

(e) $x_t \; \bar{q} \; \mu$ iff $x_t \in \mu'$.

(f) $\mu \; \bar{q} \; \mu'$.

(g) $\mu \subseteq \rho$ iff for all $x_t \in S(X)$, $x_t \; q \; \mu$ implies that $x_t \; q \; \rho$.

(h) $x_t \; q \bigcup_{j \in J} \mu_j$ iff there exists $j \in J$ such that $x_t \; q \; \mu_j$.

(i) $x_t \; q \bigcap_{j \in J}^{n} \mu_j$ iff $x_t \; q \; \mu_j$ for all $j \in \{1,2,\dots,n\}$.

(j) $x_t \; q \bigcap_{j \in J} \mu_j$ iff $x_t \; q \; \mu_j$ for all $j \in J$.

In the following denote by $\underline{\alpha}$ the constant mapping with value $\alpha \in (0,]$ and 1_A the characteristic mapping of $A \subseteq X$.

Proposition 0.1.7 [52]. Let $\mu, , \eta \in I^X$ and $\{\mu_j : j \in J\} \subseteq I^X$. Then

(a) The weak α-cuts form a descending chain, that is

$0 < \alpha_1 \leq \alpha_2 \leq 1$ implies $\mu_{W\alpha_2} \subseteq \mu_{W\alpha_1}$

(b) The strong α-cuts form a descending chain, that is

$0 < \alpha_1 \leq \alpha_2 \leq 1$ implies $\mu_{s\alpha_2} \subseteq \mu_{s\alpha_1}$

(c) Decomposition of a fuzzy set μ in terms of its strong α-cuts is

$\mu = \bigcup \{\underline{\alpha} \cap 1_{\mu_{s\alpha}} : \alpha \in [0,1)\}$, where

$$\underline{\alpha} \cap 1_{\mu_{s\alpha}} : X \longrightarrow I : x \longrightarrow \begin{cases} \alpha & ; \mu(x) > \alpha \\ 0 & ; \text{elsewhere.} \end{cases}$$

(d) $\mu \leq \eta$ iff $\mu_{s\alpha} \subseteq \eta_{s\alpha}$ for all $\alpha \in [0,1)$.

(e) $\mu \leq \eta$ iff $\mu_{w\alpha} \subseteq \eta_{w\alpha}$ for all $\alpha \in (0,1]$.

(f) $\mu = \eta$ iff $\mu_{w\alpha} = \eta_{w\alpha}$ for all $\alpha \in (0,1]$ and $\mu = \eta$ iff $\mu_{s\alpha} = \eta_{s\alpha}$ for all $\alpha \in [0,1)$.

Definition 0.1.8 [15,77]. Let X and Y be sets and $f : X \longrightarrow Y$. Then

(a) For any $\mu \in I^X$, the image $f(\mu) \in I^Y$ is defined by

$$f(\mu)(y) = \begin{cases} \sup_{x \in f^{-1}(y)} \mu(x) & \text{if } f^{-1}(y) \neq \emptyset \\ 0 & \text{elsewhere} \end{cases}$$

for all $y \in Y$.

(b) For any $\mu \in I^Y$, the inverse image $f^{-1}(\mu)$ of μ is an element of I^X and it defined by $f^{-1}(\mu)(x) = \mu(f(x))$ for all $x \in X$.

Proposition 0.1.9 [15,74,83]. Let $f : X \longrightarrow Y$, $\mu \in I^X$, $\eta \in I^Y$ and $\{\mu_j : j \in J\} \subseteq I^X$. Then

(a) $\mu_1 \leq \mu_2$ implies $f(\mu_1) \leq f(\mu_2)$.

(b) $\eta_1 \leq \eta_2$ implies $f^{-1}(\eta_1) \leq f^{-1}(\eta_2)$.

(c) $f(\mu') \geq (f(\mu))' \cap \text{rang}(f)$ and equality holds if f is bijection.

(d) $f^{-1}(\eta') = (f^{-1}(\eta))'$.

(e) $f(f^{-1}(\eta) \cap \mu) = \eta \cap f(\mu)$.

(f) $ff^{-1}(\eta) \leq \eta$ and equality holds if f is surjective.

(g) $f^{-1}f(\mu) \geq \mu$ and equality holds if f is injective.

(h) for all $x_t \in S(X)$, $f(x_t)=(f(x))_t$.

(i) $\mu_1 \; q \; \mu_2$ implies $f(\mu_1) \; q \; f(\mu_2)$.

(j) $f(\mu) \; q \; \eta$ implies $\mu \; q \; f^{-1}(\eta)$. In particular, $f(x_t) \; q \; \eta$ implies that $x_t \; q \; f^{-1}(\eta)$.

Proposition 0.1.10 [27,52]. Let $f : X \longrightarrow Y$ be a mapping, $\mu \in I^X$, $\eta \in I^Y$, $A \subseteq X$ and $B \subseteq Y$. Then

(a) $f^{-1}(\eta_{S\alpha}) = (f^{-1}(\eta))_{S\alpha}$, $\alpha \in [0,1)$.

(b) $f^{-1}(\eta_{W\alpha}) = (f^{-1}(\eta))_{W\alpha}$, $\alpha \in (0,1]$.

(c) $f(\mu_{S\alpha}) = (f(\mu))_{S\alpha}$, $\alpha \in [0,1)$.

(d) $f(\mu_{W\alpha}) \subseteq (f(\mu))_{W\alpha}$, $\alpha \in (0,1]$.

(e) $f^{-1}(1_B) = 1_{f^{-1}(B)}$ and $f(1_A) = 1_{f(A)}$.

(f) $f^{-1}(\underline{\alpha} \cap 1_B) = \underline{\alpha} \cap 1_{f^{-1}(B)}$, $\alpha \in [0,1]$.

0.2. Fuzzy topology.

Definition 0.2.1 [15]. A collection $\tau \subseteq I^X$ is said to be a fuzzy topology on X if it satisfies the following conditions

(O_1) $X, \emptyset \in \tau$.

(O_2) If $\mu, \eta \in \tau$, then $\mu \cap \eta \in \tau$.

(O_3) If $\{\mu_j : j \in J\} \subseteq \tau$, then $\bigcup_{j \in J} \mu_j \in \tau$.

The pair (X,τ) is called a fuzzy topological space (fts, for short). The members of τ are called fuzzy open sets and their complements are called fuzzy closed sets.

Note that if the family $\{\underline{\alpha} : \alpha \in [0,1]\} \subseteq \tau$; (that is all the constant mappings are open), then (X,τ) is called Lowen fts or stratified fts [57]. It is called purely stratified iff $\tau \subseteq \{\underline{\alpha} : \alpha \in [0,1]\}$ and simply stratified iff $\tau = \{\underline{\alpha} : \alpha \in [0,1]\}$ [73]. In the present thesis we do not require that (X,τ) is Lowen fts, that is we begin with the original definition of Chang [15].

Definition 0.2.2 [69]. A fuzzy set μ is called Q-neighbourhood (Q-nbd, for short)(resp. nbd) of a fuzzy point x_t iff there exists $\eta \in \tau$ such that $x_t \, q \, \eta \leq \mu$ (resp. $x_t \in \eta \leq \mu$). If μ is open then it is called an Q-open nbd (resp. open nbd). The class of all open nbd of x_t will be denoted by $N(x_t,\tau)$.

Definition 0.2.3 [35]. Let (X,τ) be a fts and $\mu \in I^X$. Then

(a) The closure of μ, denoted by $cl(\mu)$, is defined as

$$cl(\mu) = \bigcap \{\lambda : \lambda' \in \tau \text{ and } \mu \leq \lambda\}.$$

(b) The interior of μ, denoted by $int(\mu)$, is defined as

$$int(\mu) = \bigcup \{\rho : \rho \in \tau \text{ and } \rho \leq \mu\}.$$

Proposition 0.2.4 [20]. Let (X,τ) be a fts, $x_t \in S(X)$ and $\mu \in I^X$. Then

(a) For all $\eta \in \tau$, $\eta \, q \, \mu$ iff $\eta \, q \, cl(\mu)$.

(b) $x_t \, q \, cl(\mu)$ iff for all $\eta \in N(x_t,\tau)$, $\eta \, q \, \mu$.

Definition 0.2.5 [62,85]. (a) Let Y be a crisp subset of X. Then (Y,τ_Y) is called a subspace of (X,τ), where τ_Y is the fuzzy topology given by

$$\tau_Y = \{Y \cap \mu : \mu \in \tau\}.$$

A subspace Y is closed (resp. open) if the crisp subset Y of X is closed (open) in X. For any fuzzy set μ in X the relative complement $\tau_Y - \mu'$ of μ with respect to Y is a fuzzy set in X given by $\tau_Y - \mu' = Y \cap \mu'$.

(b) A subfamily β of τ is a base for τ iff each member of τ can be expressed as the union of some members of β.

(c) A subfamily ξ of τ is a subbase for τ iff the family of all finite intersections of members of ξ form a base for τ.

(d) Let $\{(X_j,\tau_j) : j \in J\}$ be a family of fts's and let $X=\prod_{j\in J} X_j$ be the cartesian product of $\{X_j : j \in J\}$. Then the product fuzzy topology τ on X is the one with basic fuzzy open sets of the form $\prod_{j\in J}\eta_j$ where $\eta_j \in \tau_j$ and $\eta_j \neq X_j$ only for a finitely many $j \in J$. The pair (X,τ) is called the product space. Note that $(\prod_{j\in J}\eta_j)(x)=\inf_{j\in J}\{\eta_j(x)\}$, where $x =(x_j)_{j\in J}$.

<u>**Definition 0.2.6 [28,79]**</u>. A mapping $f : (X,\tau) \longrightarrow (Y,\sigma)$ is called

(a) Fuzzy continuous (F-continuous, for short) iff $f^{-1}(\mu) \in \tau$ for all $\mu \in \sigma$.

(b) f is called F-open (resp. F-closed) if the image of every F-open (resp. F-closed) set in X is an F-open (resp. F-closed) in Y.

<u>**Definition 0.2.7 [28]**</u>. The mapping $f : (X,\tau) \longrightarrow (Y,\sigma)$ is said to be F-homeomorphism if it is bijective, F-continuous and f^{-1} is F-continuous. Two fts's X and Y are said to be F-homeomorphic if there is an F-homeomorphism mapping $f : X \longrightarrow Y$.

<u>**Definition 0.2.8 [28]**</u>. A property $\mathfrak{P}$ of an fts X is called a fuzzy topological property if whenever it is true for an fts X is also true for every F-homeomorphic of X.

<u>**Definition 0.2.9 [30]**</u>. (a) The fuzzy unit interval I_f is the set of all decreasing mappings $\lambda : \mathbf{R} \longrightarrow I$, ($\mathbf{R}$, the real number), for which

$$\lambda(t) = \begin{cases} 1 & \text{if } t < o \\ 0 & \text{if } t > 1. \end{cases}$$

Note that λ_1, λ_2 are equivalent if for all $t \in \mathbf{R}$

$$\lambda_1(t-) = \lambda_2(t-) \text{ and } \lambda_1(t+) = \lambda_2(t+).$$

(b) For each $t \in (0,1)$, we define

$R_t : I_f \longrightarrow I$ by $R_t(\lambda) = \lambda(t+)$ and

$L_t : I_f \longrightarrow I$ by $L_t(\lambda) = 1 - \lambda(t-)$.

(c) The fuzzy topology on I_f is that which has $\{R_t, L_t : t \in \mathbf{R}\}$ as a subbase. This topology is called the usual fuzzy topology on I_f.

Definition 0.2.10 [38,42] A fts (X,τ) is called

(a) FT_0 iff $x_t \bar{q} y_r$ implies that there exists $\mu \in \tau$ such that $(x_t \in \mu$ and $\mu \bar{q} y_r)$ or $(y_r \in \mu$ and $\mu \bar{q} x_t)$.

(b) FT_1 iff $x_t \bar{q} y_r$ implies that there exist $\mu \in N(x_t,\tau)$ and $\rho \in N(y_r,\tau)$ such that $(x_t \in \mu$ and $\mu \bar{q} y_r)$ and $(y_r \in \rho$ and $\rho \bar{q} x_t)$.

(c) FT_2 or F-Hausdorff iff $x_t \bar{q} y_r$ implies that there exist $\mu \in N(x_t,\tau)$ and $\rho \in N(y_r,\tau)$ such that $\mu \bar{q} \rho$.

(d) $FT_{2\frac{1}{2}}$ or F-Urysohn iff $x_t \bar{q} y_r$ implies that there exist $\mu \in N(x_t,\tau)$ and $\rho \in N(y_r,\tau)$ such that $cl(\mu) \bar{q} cl(\rho)$.

(e) FR_2 or F-regular iff $x_t \bar{q} \lambda$, λ is τ-closed fuzzy set implies that there exist $\mu \in N(x_t,\tau)$ and $\rho \in N(\lambda,\tau)$ such that $\mu \bar{q} \rho$.

(f) FR_3 or F-normal iff $\lambda_1 \bar{q} \lambda_2$, λ_1 and λ_2 are τ-closed fuzzy sets implies that there exist $\mu \in N(\lambda_1,\tau)$ and $\rho \in N(\lambda_2,\tau)$ such that $\mu \bar{q} \rho$.

(g) $FR_{2\frac{1}{2}}$ or F-completely regular iff $x_t \bar{q} \lambda$, λ is τ-closed fuzzy set, there exists an F-continuous mapping $f : X \longrightarrow I_f$ such that

$$x_t(y) \le f(y)(1-) \le f(y)(0+) \le \lambda'(y), \text{ for all } y \in X.$$

(h) FT_{k+1} iff it is FT_1 and FR_k, k=2,3.

Definition 0.2.11 [20]. Let (X,τ) be a fts and $\mu \in I^X$. Then

(a) A family $\{\eta_j : j \in J\} \subseteq \tau$ is called an open cover of μ iff for all $x_t \in \mu$ there exists $j \in J$ such that $x_t \in \eta_j$.

(b) μ is called a C-set iff every open cover of μ has a finite subcover.

(c) (X,τ) is called F-compact iff every closed fuzzy set is C-set.

Theorem 0.2.12 [20]. (a) Every C-set in a FT_2 fts is a fuzzy closed set.

(b) Every F-continuous image of a C-set is a C-set.

Definition 0.2.13 [86]. Let (X,τ) be a fts, $\alpha \in (0,1]$ and $\mu \in I^X$. Then

(a) The family $\Gamma \subseteq \tau$ is called an open Q_α-cover of μ iff for all $x_t \in \mu$, $t \geq \alpha$ there exists $\eta \in \Gamma$ such that $x_t \, q \, \eta$.

(b) A subfamily β of an open Q_α-cover Γ of μ which is also an open Q_α-cover of μ is called an open Q_α-subcover.

(c) μ is called $F.Q_\alpha$-compact iff each an open Q_α-cover of μ has a finite open Q_α-cover.

(d) (X,τ) is called $F.Q_\alpha$-compact iff X is $F.Q_\alpha$-compact.

Definition 0.2.14 [25]. Let (X,τ) be a fts and let $\alpha \in [0,1]$. A collection $\Gamma \subseteq \tau$ is called an α-shading (resp. α^*-shading) of X iff for each $x \in X$, there exists $\eta \in \Gamma$ with $\eta(x) > \alpha$ (resp. $\eta(x) \geq \alpha$). A subcollection β of an α-shading (resp. α^*-shading) Γ of X that is also an α-shading (resp. α^*-shading) is called an α-subshading (resp. α^*-subshading) of Γ. (X,τ) is called α-compact (resp. α^*-compact) iff each α-shading (resp. α^*-shading) of X has a finite α-subshading (resp. α^*-subshading).

Definition 0.2.15 [57]. Let (X,T) be an ordinary topological space. The set of all lower semicontinuous functions from (X,T) into the closed unit interval equipped with the usual topology constitutes, a fuzzy topology on X and it is called the induced fuzzy topology associated with (X,T) and is denoted by $(X,\omega(T))$.

Lemma 0.2.16 [73]. (a) The set $\omega_\alpha(T) = \{\mu \in I^X : \mu_{s\alpha} \in T\}$ is a fuzzy topology on X and it is called α-topologically generated fuzzy topology.

(b) $\omega(T) = (\bigcap_{\alpha\in(0,1)} \omega_\alpha(T)) = \{\mu \in I^X : \mu_{s\alpha} \in T \text{ for all } \alpha \in [0,1)\}$.

Definition 0.2.17 [59]. A property $\mathfrak{P}_f$ of a fts is said to be a good extension of the property $\mathfrak{P}$ in classical topology iff whenever the fts is topologically generated (induced) by (X,T), then $(X,\omega(T))$ has the property $\mathfrak{P}_f$ iff (X,T) has the property $\mathfrak{P}$.

Lemma 0.2.18 [27]. Let (X,T) be an ordinary topological space, $\mu \in I^X$ and and $A \in P(X)$, the power set of X. Then we have

(a) $\mu \in \omega(T)$ iff for all $\alpha \in [0,1)$ $\mu_{s\alpha} \in T$.

(b) μ is $\omega(T)$-closed iff for all $\alpha \in (0,1]$, $\mu_{w\alpha}$ is T-closed.

(c) $A \in T$ iff $1_A \in \omega(T)$.

(d) A is T-closed iff 1_A is $\omega(T)$-closed.

(e) $cl(1_A) = 1_{Cl(A)}$.

Theorem 0.2.19 [20]. Let (X,T) be an ordinary topological space. Then (X.T) is T_2 iff $(X,\omega(T))$ is FT_2.

Definition 0.2.20 [57]. Let (X,τ) be a fts. Then the family $\iota_\alpha(\tau)=\{\mu_{s\alpha} : \mu \in \tau\}$, $\alpha \in [0,1)$ is a topology on X and it constitutes a subbase for the so-called initial topology associated with (X,τ), which is denoted by $(X,\iota(\tau))$.

Lemma 0.2.21 [57]. Let (X,τ) be a fts and $\mu \in I^X$. Then :

(a) $\mu \in \tau$ iff for all $\alpha \in [0,1)$ $\mu_{s\alpha} \in \iota(\tau)$.

(b) μ is τ-closed iff for all $\alpha \in (0.1]$, $\mu_{w\alpha}$ is $\iota(\tau)$-closed.

Let $\mathfrak{F}(X)$, $\mathfrak{P}(X)$ be the collections of all filters, respectively prefilters on X [73]. Also $\mathfrak{P}^c(X)$ denotes the set of all prefilters on X which excluded the constant functions $\underline{\alpha}$, that is $\beta \in \mathfrak{P}^c(X)$ iff $\beta \in \mathfrak{P}(X)$ and $\underline{\alpha} \notin \beta$. The elements of $\mathfrak{P}^c(X)$ are called α-prefilters. Associated with each α-prefilter β, we define a filter on X by, $\iota_\alpha(\beta) = \{\mu_{s\alpha} : \mu \in \beta\}$.

Likewise, associated with each filter Γ on X, a α-prefilter for each $\alpha \in [0,1)$ is defined by $\omega_\alpha(\Gamma) = \{\mu \in I^X : \mu_{s\alpha} \in \Gamma\}$.

Definition 0.2.22 [73]. Let (X,τ) be a fts, $\mathfrak{F} \in \mathfrak{P}(X)$ and $x_\alpha \in S(X)$, $\alpha \in [0,1)$. We say that $\mathfrak{F}$ α-converges to x_α (or $\mathfrak{F} \xrightarrow{\alpha} x_\alpha$) if

(a) $\mathfrak{F} \in \mathfrak{P}^c(X)$.

(b) The filter $\iota_\alpha(\mathfrak{F})$ on X convergs to x in the topological space $(X,\iota_\alpha(\tau))$.

Proposition 0.2.23 [73]. (a) Let (X,τ) be a fts and $Y \subseteq X$. Then $\iota_\alpha(\tau_Y) = (\iota_\alpha(\tau))_Y$.

(b) Let $\{(X_j,\tau_j) : j \in J\}$ be a family of fts's. Then $\iota_\alpha(\prod_{j\in J} \tau_j) = \prod_{j\in J} \iota_\alpha(\tau_j)$.

Definition 0.2.24 [73]. A fts (X,τ) is called

(a) α-topologically generated (or α.TG) if there exists a topology T on X such that $\omega_\alpha(T) = \tau$.

(b) Topologically generated (or TG) if there exists a topology T on X such that $\omega(T) = \tau$.

Note that, if (X,τ) is α.TG (resp. TG), then