INTRODUCTION

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Different approaches have been presented for finding invariants of some nonlinear partial differential equations (PDEs). In these approaches, invariants are found by using free parameter method [1], separation of variables method [2], group properties for differential equations [3,4], and dimensional analysis for finding the invariants [5].

Obtaining invariant solutions reduce to solving quotient differential equations in fewer independent variables than the original equations. In particular, These quotient equations might be ordinary differential equations (ODEs).

The motivation of this work is to apply a method called the similarity method not only to find invariants of nonilinear PDEs but also to predict the existance of invariants (this method essentially dates back to the original investigations of ***Sophus Lie*** [6-8]). By this method, one can try to obtain invariant, partially invariant solution to perform the group, to transform a given PDE to a less complicated or ODE via one-parameter Lie group of transformations [4,5,9].

This method is used to the nonlinear PDEs which describe some nonlinear physical problems in order to construct all possible classes of similarity solutions and to give a strong group-theoretical classification of the results. These classes of solutions can be obtained and classified by means of the similarity method if the group constants allowed to take some special values. Besides these known classes of solutions the method is shown to create additional classes of similarity solutions heretofore undiscovered. We wish to remark that the similarity method may be called to play an interesting role in nonlinear mechanics. For instance in the article .[10] the complete nonlinear problem of water wave is investigated according to the perfect fluid model, using general methods of infinitesimal transformations theory for finding the symmetry group of free-boundary problems.

The application of Lie group theory to obtain exact analytical solutions (the so-called invariant solutions) of a PDE is widely known [5,6].

The similarity reduction of a PDE is the procedure by which it is possible to transform an equation having n independent variables to another in which only n-1 independent variables appear. It is obtained by means of a suitable choice of the variables and the unknown functions. Hence, for n=2, the similarity reduction transforms a PDE into an ordinary DE and makes possible the determination of a class of solutions of the given PDE depending on arbitrary constants.

There exist two main approaches to the problem of finding similarity reduction of a given PDE

(I) The Lie approach.

Lie **[4,5,9]** considered groups of point transformations depending on continuous parameters, acting on the space of independent and dependent variables of a given PDE. Such a group is completely characterized in terms of infinitesimal functions which depend on the independent and dependent variables of the given PDE.

Lie extended his work to groups of contact transformations that act on the space of independent and dependent variables and first derivatives of the dependent variables of the given DE **[4,5,9]**.

The basic idea underlying Lie's approach is to study the invariance properties of the given DEs under continuous groups of transformations. If the most extensive Lie groups of transformations admitted by a given DE, is known, then it is possible to construct classes of particular solutions, called the similarity solutions of the PDE.

The symmetry reduction of the PDE can be classified in two categories :

(1) The classical method (conventional Lie theory)

The classical method to obtain similarity reduction is to use the symmetry properties of the equation : for any group of point symmetries admitted by the equation, it is possible to obtain the necessary and sufficient condition for invariance of the PDE with respect to the one-parameter group of transformations. Setting all the coefficients of like derivatives equal zero, one obtains a system of determining equations for the group elements. The group of point symmetries which leaves the PDE invariant may be determined by means of group elements of its generators. These infinitesimal generators must form a Lie algebra determined by the structure constants. This means that the set of generators chosen must be closed under the commutation operator.

Having defined the generators of a symmetry group, the invariance relation allows determination of the similarity variable and the form of the similarity solution, using the general integral of the characteristic system.

The symmetry transformation group can be generated from the corresponding Lie algebra. In fact to find the group we only have to find its Lie algebra. We will seek the Lie algebra and determine its corresponding transformation group.

In general, to each subgroup of the full group corresponds a family of invariant solutions. In addition, there is always an infinite number of such subgroups which can be classified into conjugate classes. Every subgroup and its invariant solutions belonging to the same class be derived from each other under the group action. So we only have to discuss one subgroup from each class. A subgroup from each class may be viewed as a representation of the class. The optimal system is the set of all representation subgroups. From this optimal system we can construct all the group invariant solutions.

To find the optimal system we have to use the adjoint representation of the full group. The adjoint representation of the Lie algebra described in [4,9]. The adjoint representation of the Lie algebra is just the Lie bracket operator. Any two one-parameter groups are in the same conjugate class if their infinitesimal generators can be transformed into each other under some action of Adj. . Symmetry groups and the corresponding invariant solutions using classical method will be consider in solving some physical problems.

(2) The non-classical method

Bluman and Cole [11] proposed a generalization of Lie's method called the non-classical method of group invariant solutions, which itself has been extended by ***Olver and Rosenau*** [12,13] who concluded that "the unifying theme behind finding special solutions of a PDE is not as is commonly suppose, group theory, but rather the more analytic subject of overdetermined system of PDEs". The class of symmetries in this method is called the class of conditional non-classical symmetries. The term "conditional" is explaind by the fact that symmetries of a new system of DEs are examined. The latter obtained appending additional DE called side condition or the invariant surface condition to the original DE.

The non-classical method involves more algebra and calculations than the classical Lie method [14,15]; in fact ***Olver and Rosenau*** **[12,13]** suggest that for some PDEs the determining equations for these non-classical method might to too difficult to solve explicitly. The principal reason for this is that although the detemining equations for the infinitesimals in the classical method are a linear system of equations for the infinitesimals, in the nonclassical method, they are a nonlinear system. For some equations, such as the linear heat equation, it is will known that the nonclassical method does not appear to yield any more similarity reductions than the classical Lie method does **[16,17]**. Several examples for finding new invariant solutions using the classical and nonclassical methods considered here.

The above two methods determine Lie point transformations of a given PDE, i.e. transformations depending on the dependent and independent variables.

The PDEs can be invariant under continuous group transformations beyond point or contact transformation Lie group, which act on a finite dimensional space **[18]**. These new continuous group transformations act on an infinite dimensional space. Such infinite dimensional transformations have been called **Noether** transformations **[19,20]** or Lie-Backlund (LB) transformations. These transformations depend upon the derivatives of the dependent variables . Well known nonlinear PDEs admitting LB transformations include the korteweg-de vries (kdv) **[21,22]**, sine-Gardon **[20,23]** and Burgers equations **[24]**.

A common characteristic of all these methods for finding symmetries and associated similarity reduction is the use of the group theory.

In **1988 Bluman and Kumei [25]** introduce a method which yields new classes of symmetries of a given PDE. They used Lie groups of point transformations whose infinitesimals act on a defferent space than the space of independent variables, dependent variables, and their derivatives of the given DE. Their new symmetries are neither point symmetries nor Lie-Backlund symmetries. This space depends on the independent variables, dependent variable and the potential after writting the PDE in a conserved form.

(II) The direct method.

In **1989 Clarkson and Kruskal [26]** developed a method for finding similarity reductions of PDEs called the direct method and using it we obtain previously unknown similarity reductions of the PDEs. The novel characteristic about this direct method is that it involves no use of group theory.

The basic idea of this method is to seek a reduction of a given PDE (with two independent variables x, t and one dependent variable u is in the form $u(x,t) = F(x,t,\eta)$ with $\eta = \eta(z)$ and $z = z(x,t)$, which is the most general form for a similarity reduction [27]. Substituting this into the PDE and demanding that result be an ordinary DE for $\eta(z)$ imposes conditions upon $F(x,t,\eta)$, $z(x,t)$ and their derivatives in the form of an overdetermined system of equations whose solution yields the similarity reduction.

The classical and nonclassical methods yields more information than the direct method since a considerable amount of similarity reductions obtained from the classical and nonclassical methods can not be found using the direct method. For example **Pucci [15]** has argued that in Bergers equation, Lie approach yields more similarity reductions than the direct method. For the Boussinesq equation **Levi and Winternhitz [31]** have shown that both two methods lead to the same similarity reductions.

The applications of the direct method **[26,28,29,30]** involved reducing DEs with two independent variables only to ordinary DEs, but Lie methods can be used to transform an equation having n independent variables to another in which only n-1 independent variables appear. Also,Lie methods can be used to a system of PDEs **[5]**.

We remark that this direct method has certain resemblances to the so-called method of free parameter analysis **[32]**. Through in the latter method, the boundary conditions are crucially used in the determination of the similarity reduction where as they are not used in the direct method.

The thesis is organized as follows :

Chapter 1, contains the basic concepts of Lie groups of transformations and Lie algebra necessary in subsequent chapters

Chapter 2

In this chapter we apply the method of group invariant solutions **[4,5,33]** to determine new similarity reductions of the one-dimensional inhomogeneous nonlinear diffusion equation

$$x^p u_t = \frac{\partial}{\partial x}(x^m u^n u_x)$$

which is of considerable importance both in physics and mathematics **[34-38]**. We determine the Lie point symmetry vector fields and determine new similarity reduction in addition to the previously known ones **[39,40]**

[The results of this chapter published in [60]].

Also in chapter 2, we used the non-classical method to find new symmetry reduction of the same equation.

Chapter 3.

We apply the Lie group to deduce the classical symmetries of a nonlinear model of heat equation

$$x^p u_t = \frac{\partial}{\partial x}(x^m u^n u_x^q) \quad .$$

and its Lie algebra is found. From the Lie algebra the optimal system of the model is constructed. After that, new classes of similarity solutions are obtained.

[The results of this chapter published in [62]].

Chapter 4

Using Lie group methods , we analyse nonlinear diffusion equations in an inhomogeneous medium

$$f(x)u_t = \frac{\partial}{\partial x}(g(x)D(u)u_x) \quad .$$

with arbitrary diffusion coefficient D(u), and arbitrary thermal coefficients f(x) and g(x), which have a wide spectrum of applications in many areas of science. The Lie group based similarity leads to a classification of the diffusion and thermal coefficients according to its symmetry properties. With the help of the adjoint reprentation, the optimal system of similarity reductions is calculated. Exact similarity solutions of the second-order ODEs resulting from the reductions are demonstrated by examples.

Chapter 5

We consider the way in which a solution to a class of nonlinear PDEs