# Introduction

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Group-theoretic methods are powerful, versatile and fundamental to the development of systematic procedures that lead to invariant solutions of differential equations. The group-theoretic methods are applicable to both linear and nonlinear differential equations since they are not based on linear operators, superposition or other requirements of the linear solution techniques.

A systematic investigation of continuous transformation group was carried by Lie [1-3]. His original goal was the creation of a theory of integration for ordinary differential equations analogous to the Abelian theory for the solution of algebraic equations. He investigated the concept of the invariance groups admitted by a given system of differential equations. These groups have important real world applications. A number of books on the application of the continuous groups of transformations relating to differential equations have been written from a mathematical standpoint, for example Bluman and Cole [4], Ovisannkov [5], Hill [6] Olver [7], Bluman and Kumei [8], Hans [9] and Ibragimov [10]. In addition, the works of Hansen [11], Ames [12,13] and Dresner [14], Na and Hensen [15] present quite extensively the general theories involved in the similarity solution of differential equations as applied to engineering and physics problems.

In this thesis, we applied a one-parameter Lie group method not only to find invariants of partial differential equations but also to predict the existence of invariants. Using this method, one can try to obtain invariant, and partially invariant solution to form the group in order to transform a given partial differential equation to a less complicated or ordinary differential equation [7,8,13].

Similarity transformations essentially reduce the number of independent variables in partial differential equations by one. Hence, for partial differential equations which have two independent variables, the similarity transformations transform a partial differential equation into an ordinary differential equation and make the determination of a class of solutions possible for the given partial differential equation depending on arbitrary constants. Similarity transformations have been also used to convert moving boundary conditions to constant boundary conditions. There has been considerable interest in symmetry reductions of partial differential equations, mainly because the procedure reduces the number of independent variables, and, therefore, assists in the determination of exact solutions.

The classical method [5,6,13,15] for obtaining the similarity reductions is using the symmetry properties of the partial differential equations. It is possible to obtain the necessary and sufficient conditions for invariance of the partial differential equations with respect to the one-parameter group of transformations. Setting all the coefficients of like derivatives equal zero, one obtains a system of determining equations for the group elements. The group of points symmetries which leaves the partial differential equations invariant may be determined by means of its generators. These infinitesimal generators must form a Lie algebra determined by the structure constants. This means that the set of generators chosen must be closed under the commutative operator. Having defined the generators of a symmetry group, the invariance reduction allows determination of the similarity variable and the form of similarity solution, using the general integral of the characteristic system. The symmetry transformation group can be generated from the corresponding Lie algebra. In fact to find the group we only have to find its Lie algebra. We will seek the Lie algebra and determine its corresponding transformation group.

Once one has determined the symmetry group of a system of differential equations, a number of applications become available. To start, one can directly use the defining property of such a group and construct new solutions to the system from known ones. The symmetry group thus provides a means of classifying different symmetry classes of solutions, where two solutions are deemed to be equivalent if one can be transformed into the other by some group element. Alternatively, one can use the symmetry groups to affect a classification of families of differential equations depending on arbitrary parameters or functions. Often there are good physical or mathematical reasons for preferring those equations with as high a degree of symmetry as possible. The classical method has been used to study a variety of equations such as Burgers' equations by Chester [16], Nonlinear Carleman-Boltzmann Equation by Dukek and Nonnenmacher [17], wave equations by Bluman and Kumei [28], one-and two-dimensional Fokker-Plank equations by Shtelen and Stogry [19], Fragmentation Kinetics Model by Baumann , Freyberger , Glokle and Nonnenmcher [20], Nonlinear heat equations by Peter Clarkson and Mansfield [21], Telford coagulation-equation by El-wakil el at [22], generalized diffusion reaction equation by Nonnemacher [23], Free Kramer equation by Saied [24], fragmentation equation by Saied and El-Wakil [25], Shallow water wave equation by Clarkson and Mansfield [26] , inhomogeneous nonlinear diffusion equation by Saied and Hussein [27], nonlinear diffusion-convection equations by Edwards [28], generalized inhomogeneous nonlinear diffusion equation by Saied [29], nonlinear model of heat equation by Saied and Hussein [30], heat conduction in metals by Saied [31], porous medium equation by Maria [32], continuity equation for electrons in micro wave-after glow plasma by Abdel – Gawad [33], Hetrogeneous Schrodinger equation by Zedan [34], generalized inhomogeneous nonlinear diffusion convection equation by Saied [35], and important physical problems in applied mathematics, physics and engineering see [6 ,7,9 ,13 ].

A generalization of the above method called non-classical method was proposed by Bluman and Cole [36]. The family of solutions is now larger than that obtained by the classical method. This technique has been further generalized by Olver and Rosenau [37]. By this method, Bluman and Cole [36] found new solutions of the heat equation which are not derivable by Lie's classical method. Levi and Winternitz [38] found classes of solutions for the Boussinesq equation not derivable by classical method, Arrigo, Hill and Broadbridge [39] found different reductions of the linear diffusion equation with a nonlinear source, Lou and Ruan [40] found all similarity reductions of the Kupershmidt equation, Saied and Hussain [26] found new solutions of the inhomogeneous nonlinear diffusion equation not derivable by classical method and Nucci [41] found some exact solutions of Fitzhugh-Nagumo equation.

These two methods determine Lie point symmetries of a given partial differential equation since the transformations dependent only on the independent and dependent variables. Another common characteristic of these methods for finding similarity reductions of a given partial differential equation is use of group theory.

A direct method which does not use group analysis techniques for determining the symmetry reductions of given partial differential equations has been developed by Clarkson and Kruskal [42]. The basic idea of this method is to seek a reduction of a given partial differential equation (with two independent variables $x$, $t$ and one dependent variable $u$) in the form

$$u(x, t) = W( x, t, F )$$

with $F(z)$ as a function, and $z = z( x,t )$ as a similarity variable . Substituting this into the partial differential equation and demanding that the result be an ordinary differential equation for $F(z)$ impose conditions upon $W(x,t,F)$, $z(x,t)$ and their derivatives in the form of an overdetermined system of equations whose solution yields the similarity reduction. Clarkson and Kruskal [42] method proved effective in framing new symmetry solutions to a variety of partial differential equations. Later, Levi and Winternitz [43] and Pucci and Succomandi [44] recognized that solutions derived by the direct method of Clarkson and Kruskal [45] are always invariant solutions under non-classical symmetries admitted by the equation. However, partial differential equations may admit symmetry reductions with non-classical symmetries, yet they are not recoverable by the direct method.

Unfortunately, for systems of partial differential equations, the symmetry group is usually of no help in determining the general solution (although in special cases it may indicate when the system can be transformed into a more easily solvable system such as a linear system). However, one can use general symmetry groups to explicitly determine special types of solutions which are themselves invariant under some subgroups of the full symmetry group of the system. These group-invariant solutions are found by solving a reduced system of differential equations involving fewer independent variables than the original system.

**The present thesis consists of five chapters:**

**Chapter 1.** In this chapter, we introduce the basic ideas of Lie group of transformations necessary for the study of invariance properties of differential equations.

**Chapter 2.** In this chapter, we apply the method of group invariant solutions to determine the different exact solutions of the nonlinear partial differential equation

$$u_t = Du_{yy} - [vLu(1-u)]_x .$$

This equation determines the concentration of the particles moving by diffusion and drift. The motion of the particles past an impenetrable obstacle in two dimensions. Where $D$ diffusion coefficient, $v$ drift velocity and $L$ length of the rod. We study qualitative features of some exact solutions

**[The result of this chapter published in [105]]**

**Chapter 3.** Using Lie group methods, we analyse the nonlinear partial differential equation

$$f(x)u_t = \left(g(x)D(u)\, u_x^{\,m}\right)_x + H(u)u_x \ , \quad m \neq 0 \ \text{or} \ 1,$$

with arbitrary diffusion coefficient $D(u)$, Conductivity coefficient $H(u)$ and variable coefficients $f(x)$ and $g(x)$, which have a wide spectrum of applications in porous medium. The Lie group based similarity leads to a classification of the diffusion, conductivity coefficients and variable coefficients according to its symmetry properties. We find the forms of coefficients ($f$, $g$, $D$, $H$) which have additional symmetries.

**[The result of this chapter published in [106]]**

**Chapter 4.** In this chapter, we study the similarity solutions of two-dimensional nonlinear Schrodinger equation

$$-iu_t + pu_{xx} + qu_{yy} + r|u|^2 u = 0 ,$$

by using Lie method of infinitesimal transformation groups. It is shown that the two-dimensional nonlinear Schrodinger equation can be reduced to the nonlinear Klein-Gordon equation and others. Some new similarity solutions of these equations are obtained.

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**Chapter 5.** In this chapter, we apply the Lie group analyses to deduce the classical symmetries of adivection-diffusion equation

$$u_t + v_0(1-\frac{y^2}{a^2})u_x = D\left[y^{1-\delta}\frac{\partial}{\partial y}(y^{\delta-1}u_y)\right],$$

which describe the transport of contaminants in a fractured medium, and its Lie algebra is found. From the Lie algebra the optimal system of this equation is constructed. After that, different analytic solutions are obtained.

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