CHAPTER 0

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In this thesis we investigate the properties of closed densely defined operators generated in a weighted Hilbert space $L^2_w(a,b)$ by $\frac{1}{w}M$, where M is a general quasi - differential expression defined by

$$M[u] = i^n u^{[n]} , \tag{0.1}$$

this being defined on the set

$$V(M) := \{ u : u^{[r-1]} \in AC_{loc}(a,b), \quad r = 1,2, \ldots ,n \} .$$

Two problems are considered, the regular problem and the singular one. In the regular case, the problem is defined on the closed interval $[a,b]$. In the singular case we consider an interval $[a,b)$ with $-\infty < a < b \leq \infty$. By a standard procedure, the singular problem on an open interval with $-\infty \leq a < b \leq \infty$, can be treated in terms of the problems on $(a,c]$ and $[c,b)$ for some $c \in (a,b)$.

Associated with M is its formal (or Lagrangian) adjoint

$$M^+[v] = i^n v_+^{[n]} , \tag{0.2}$$

this being defined on the set

$$V(M^+) := \{ v : v_+^{[r-1]} \in AC_{loc}(a,b), \quad r = 1,2, \ldots ,n \},$$

which also generates operators in $L^2_w(a,b)$. If M is formally symmetric, i.e., $M = M^+$, then the basic operators considered (the so - called minimal operators) are symmetric and the celebrated Stone - von - Neumann theory applies to give a complete characterization of all the closed symmetric extensions.

For the general quasi - differential expression M considered here, the minimal operator is no longer symmetric. However, the minimal operators $T_o(M)$, $T_o(M^+)$ generated by M and M^+ respectively do form an adjoint pair in the sense defined in the definitions below. This enables us to apply the theory developed by W.D.Evans in [12] (see also the book [11] by Edmunds/ Evans) concerning the so - called extensions of $T_o(M)$ which are regularly solvable and well - posed with respect to $T_o(M)$ and $T_o(M^+)$. A complete characterization of these operators is given in terms of boundary conditions to be satisfied by functions in their domains. Also, their spectral properties are investigated in detail. Special cases are self - adjoint operators and J - self - adjoint operators with non - empty field of regularity, where J denotes complex conjugation.

To assist in the reading, we now define the terms which we repeatedly use throughout the thesis and state some definitions from [1], [2], [9], [24] and [29] and also some of the main results from [11] which we need. We denote by H a Hilbert space over $\mathbb{C}$, with inner product $(.,.)_H$ and norm $\|.\|_H$.

Definition 0.1 : The operator T in H is said to be closed, if its graph

$$G(T) = \{ \{ \phi , T\phi \} , \phi \in D(T) \},$$

is a closed subspace of H x H with the graph norm,

$$\| \{ \phi , \psi \}_H \| = \{ \| \phi \|_H^2 + \| \psi \|_H^2 \}^{\frac{1}{2}} ,$$

where D(T) denotes the domain of T. Equivalently, T is closed if and only if either of the following is satisfied :

(1) If $\phi_n \in D(T)$, $n = 1,2, \ldots$ are such that, as $n \longrightarrow \infty$, $\phi_n \longrightarrow \phi$ and $T\phi_n \longrightarrow \psi$ in H, then $\phi \in D(T)$ and $\psi = T\phi$.

(2) D(T) with the graph inner product $(\phi,\psi)_T$, is a Hilbert space, where,

$$(\phi,\psi)_T = (\phi,\psi)_H + (T\phi,T\psi)_H .$$

The set of closed operators in H will be denoted by C(H) .

Definition 0.2 : The set of complex numbers λ, for which there exists a positive constant $K(\lambda)$ such that, for all $\phi \in D(T)$,

$$\| (T - \lambda I)\phi \|_H \geq K(\lambda) \| \phi \|_H ,$$

is called the field of regularity of the operator T and is denoted by $\Pi(T)$. From this definition, we note that $\lambda \in \Pi(T)$ if and only if $(T - \lambda I)^{-1}$ exists and is bounded on its domain of definition $R(T - \lambda I)$ (which denotes the range of the operator $(T - \lambda I)$); $R(T - \lambda I)$ is the set of functions ψ such that $(T - \lambda I)\phi = \psi$, where $\phi \in D(T)$. If T is closed, $\lambda \in \Pi(T)$ if and only if $(T - \lambda I)^{-1}$ exists and $R(T - \lambda I)$ is closed.

Definition 0.3 : Any two operators A,B in H are said to form an adjoint pair if for all $x \in D(A)$ and $y \in D(B)$

$$(Ax,y)_H = (x,By)_H .$$

If A,B are densely defined, A,B are an adjoint pair if and only if

$A \subset B^*$ and $B \subset A^*$.

Definition 0.4 : The deficiency index of the operator $(A - \lambda I)$, written $\mathrm{def}(A - \lambda I)$, is defined to be co – dimension of $R(A - \lambda I)$; if $R(A - \lambda I)$ is closed,

$$\mathrm{def}(A - \lambda I) = \dim R(A - \lambda I)^{\perp} .$$

Definition 0.5 : We denote by $\Pi(A,B)$ the set of complex numbers λ which are such that $\lambda \in \Pi(A)$, $\bar{\lambda} \in \Pi(B)$ and both $\mathrm{def}(A - \lambda I)$ and $\mathrm{def}(B - \bar{\lambda} I)$ are finite. The set $\Pi(A,B)$ is called the joint field of regularity of A and B. An adjoint pair of closed densely defined operators A,B is said to be compatible if the joint field of regularity $\Pi(A,B)$ is not the empty set.

Definition 0.6 : The nullity of an operator T in H, written nul(T), is the dimension of the null space

$$N(T) = \{ \phi : T\phi = 0 \} .$$

A closed operator T is said to be Fredholm if its range R(T) is closed and both nul(T) and def(T) are finite. The Fredholm domain of T is the set

$$\Delta_3(T) = \{ \lambda \in \mathbb{C} : (T - \lambda I) \text{ is Fredholm} \} .$$

The number $\mathrm{ind}(T - \lambda I) = \mathrm{nul}(T - \lambda I) - \mathrm{def}(T - \lambda I)$ is called the index of $(T - \lambda I)$. The set of Fredholm operators in H is denoted by F(H).

Theorem 0.7 : ([11, Theorem I.3.2])

Suppose that T is a closed linear operator in H and that $\mathrm{def}(T) < \infty$. Then range R(T) is closed.

Definition 0.8 : The resolvent set $\rho(T)$ of a closed operator T in H consists of the complex numbers λ for which $(T - \lambda I)^{-1}$ exists, is defined on H and is bounded. The complement of $\rho(T)$ in $\mathbb{C}$ is called the spectrum of T and written $\sigma(T)$. The point spectrum $\sigma_p(T)$, continuous spectrum $\sigma_c(T)$ and residual spectrum $\sigma_r(T)$ are the following disjoint subsets of $\sigma(T)$:

$\sigma_p(T) = \{ \lambda \in \sigma(T): (T - \lambda I)$ is not injective $\}$, i.e., the set of eigenvalues of T,

$\sigma_c(T) = \{ \lambda \in \sigma(T): (T - \lambda I)$ is injective, $R(T - \lambda I) \subsetneq \overline{R(T - \lambda I)} = H\}$,

$\sigma_r(T) = \{ \lambda \in \sigma(T): (T - \lambda I)$ is injective, $\overline{R(T - \lambda I)} \neq H \}$.

For a closed operator T we have

$$\sigma(T) = \sigma_p(T) \cup \sigma_c(T) \cup \sigma_r(T) .$$

Definition 0.9 : Let T be a bounded linear operator in H and suppose that $\lambda \in \sigma_p(T)$. The geometric multiplicity of λ is defined to be $\mathrm{nul}(T - \lambda I)$.

Definition 0.10 : Let T be a bounded linear operator in H and suppose that $\lambda \in \sigma_p(T)$. The linear subspace M_λ of H defined by

$$M_\lambda = \{ x \in H : (T - \lambda I)^n x = 0 \text{ for some } n \in \mathbb{N} \} = \bigcup_{n=1}^{\infty} \mathrm{Ker}(T - \lambda I)^n$$

is called the algebraic (or root) eigenspace corresponding to λ, and non - zero element of its are called generalized eigenvectors (or root vectors) corresponding to λ. The dimension of M_λ is called the algebraic multiplicity of λ. The algebraic multiplicity of an eigenvalue is greater than or equal to its geometric multiplicity.

Definition 0.11 : A closed operator S in H, is said to be regularly solvable with respect to a compatible adjoint pair A,B of closed densely defined operators if, $A \subset S \subset B^{*}$ and $\Pi(A,B) \cap \Delta_4(S) \neq \emptyset$, where,

$$\Delta_4(S) = \{ \lambda : \lambda \in \Delta_3(S), \ \ \text{ind}(S - \lambda I) = 0 \}.$$

If $A \subset S \subset B^{*}$ and the resolvent set $\rho(S)$ of S is non - empty, S is said to be well - posed with respect to A,B. Note that if $A \subset S \subset B^{*}$ and $\lambda \in \rho(S)$ then $\lambda \in \Pi(A)$ and $\bar{\lambda} \in \rho(S^{*}) \subset \Pi(B)$ since $B \subset S^{*} \subset A^{*}$. Thus, if def$(A - \lambda I)$ and def$(B - \bar{\lambda} I)$ are finite, then A and B are compatible and S is regularly solvable with respect to A,B. The terminology " regularly solvable " comes from Visik's paper [31], while the notion of " well - posed " was introduced by Zhikhar in his work on J - self - adjoint operators in [38].

Definition 0.12 : Let M and $\tilde{M}$ be two linear manifolds such that $M \subset \tilde{M}$. The vectors $f_1, f_2, \ldots, f_n$ of $\tilde{M}$ are called linearly independent modulo M if from,

$$\alpha_1 f_1 + \alpha_2 f_2 + \ldots + \alpha_n f_n \in M,$$

it follows that

$$\alpha_1 = \alpha_2 = \ldots = \alpha_n = 0.$$

It is evident that a set of vectors in $\tilde{M}$ which are linearly independent modulo M is also linearly independent in the ordinary sense.

The dimension of $\tilde{M}$ modulo M we define as the maximal number n of vectors in $\tilde{M}$ which are linearly independent modulo M.

The dimension of $\tilde{M}$ modulo M defined above is the ordinary dimension of the quotient manifold $\{ \tilde{M}/M \}$.

We shall also need the following results from [11].

<u>Theorem 0.13</u> : ([11, Theorem III.3.1])

Let $T \in C(H)$ be densely defined and $\Pi(T) \neq \emptyset$. Then for any closed extension S of T and $\lambda \in \Pi(T)$,

$$D(S) = D(T) \dotplus N([T^* - \bar{\lambda}I][S - \lambda I]).$$

If there exists a $\lambda \in \Pi(T)$ such that, $\mathrm{def}(T - \lambda I) < \infty$ and $\lambda \in \Delta_3(S)$, then

$$\dim\{ D(S)/D(T) \} = \mathrm{nul}(S - \lambda I) + \mathrm{def}(T - \lambda I) - \mathrm{def}(S - \lambda I).$$

<u>Theorem 0.14</u> : ([11, Corollary III.3.2])

Let A,B be closed densely defined operators in H, which form an adjoint pair. Suppose there exists a $\lambda \in \Pi(A)$ such that $\bar{\lambda} \in \Pi(B)$. Then

$$D(B^*) = D(A) \dotplus N([A^* - \bar{\lambda}I][B^* - \lambda I]),$$

$$D(A^*) = D(B) \dotplus N([B^* - \lambda I][A^* - \bar{\lambda}I]),$$

and if, $\mathrm{def}(A - \lambda I)$ and $\mathrm{def}(B - \bar{\lambda}I)$ are finite,

$$\dim\{ D(B^*)/D(A) \} = \dim\{ D(A^*)/D(B) \}$$

$$= \mathrm{def}(A - \lambda I) + \mathrm{def}(B - \bar{\lambda}I).$$

Thus, $\operatorname{def}(A - \lambda I) + \operatorname{def}(B - \bar{\lambda} I)$ is constant for all $\lambda \in \Pi(A,B)$.

Theorem 0.15 : ([11, Theorem III.3.3]; this result is due to Visik [31])

Let A,B be closed densely defined operators in H, which form an adjoint pair and suppose there exists a $\lambda \in \Pi(A)$ such that, $\bar{\lambda} \in \Pi(B)$. Then, there exists a closed operator S such that, $A \subset S \subset B^*$ and $\lambda \in \rho(S)$; hence S is well - posed with respect to A and B.

Theorem 0.16 : ([11, Theorem III.4.6]; this result is due to Calkin [4])

If T is a closed symmetric operator and $\lambda \notin \mathbb{R}$, there exists a closed operator S such that $T \subset S \subset T^*$ and $\lambda \in \rho(S)$. If there exists a real $\lambda \in \Pi(T)$, then there exists a self - adjoint extension S of T with $\lambda \in \rho(S)$.

Theorem 0.17 : ([11, Theorem III.3.5])

Let S be regularly solvable with respect to the compatible adjoint pair A,B. Then, if $\lambda \in \Pi(A,B) \cap \Delta_4(S)$

$$D(S) = D(A) \dotplus N([A^* - \bar{\lambda} I][S - \lambda I]) ,$$

$$D(S^*) = D(B) \dotplus N([B^* - \lambda I][S^* - \bar{\lambda} I]) ,$$

and

$$\dim\{ D(S)/D(A) \} = \operatorname{def}(A - \lambda I),$$

$$\dim\{ D(S^*)/D(B) \} = \operatorname{def}(B - \bar{\lambda} I).$$

Theorem 0.18 : ([11, Theorem III.3.6])

If S is regularly solvable with respect to the compatible adjoint pair A,B and $\lambda \in \Pi(A,B) \cap \Delta_4(S)$, then

$$D(S) = \left\{ u : u \in D(B^*) \text{ and } \beta[u,\phi] = 0, \text{ for all } \phi \in N([B^* - \lambda I][S^* - \bar{\lambda} I]) \right\},$$

$$D(S^*) = \left\{ v : v \in D(A^*) \text{ and } \beta[\psi,v] = 0, \text{ for all } \psi \in N([A^* - \bar{\lambda} I][S - \lambda I]) \right\},$$

where

$$\beta[u,v] = (B^*u,v) - (u,A^*v), \qquad u \in D(B^*),\ v \in D(A^*).$$

Definition 0.19 : An operator J defined on a Hilbert space H is a conjugation operator if, for all x,y ∈ H,

$$(Jx,Jy) = (y,x), \qquad J^2 x = x .$$

A simple example in any L^2 - space is the complex conjugation $x \longrightarrow \bar{x}$.

The definition implies that a conjugation J is a conjugate linear, norm - preserving bijection on H and that

$$(Jx,y) = (Jy,x), \qquad \text{for all } x,y \in H.$$

Furthermore, if T is a densely defined linear operator in H, then

$$(JTJ)^* = JT^*J.$$

Definition 0.20 : A densely defined linear operator T in H is said to be J - symmetric , for a conjugation operator J, if

$$JTJ \subset T^*.$$

T is said to be J - self - adjoint if,

$$JTJ = T^*.$$

If $JTJ = T$, the operator T is said to be real with respect to J.

Definition 0.21 : (Essential spectra)

An important subset of the spectrum of a closed densely operator T in H is the so - called essential spectrum . However, there are various essential spectra for a closed operator T in the literature and these are different in general. The main ones are defined as follows. First, let

$$\Phi_+(T) = \{ \lambda \in \mathbb{C} : R(T - \lambda I) \text{ closed and } \mathrm{nul}(T - \lambda I) < \infty \},$$

$$\Phi_-(T) = \{ \lambda \in \mathbb{C} : R(T - \lambda I) \text{ closed and } \mathrm{def}(T - \lambda I) < \infty \},$$

$$\Delta_1(T) = \Phi_+(T) \cup \Phi_-(T),$$

$$\Delta_2(T) = \Phi_+(T),$$

$$\Delta_3(T) = \Phi_+(T) \cap \Phi_-(T),$$

$$\Delta_4(T) = \{ \lambda \in \mathbb{C} : \lambda \in \Delta_3(T) \text{ and } \mathrm{ind}(T - \lambda I) = 0 \},$$

$$\Delta_5(T) = \text{union of all the components of } \Delta_1(T) \text{ which intersect } \rho(T).$$

Then the essential spectra of T are the sets

$$\sigma_{ek}(T) = \mathbb{C} \setminus \Delta_k(T), \qquad k = 1,2,3,4,5 . \tag{0.4}$$

The sets $\sigma_{ek}(T)$ are closed since $\Delta_k(T)$ are open as proved in

[11, Theorems I.3.18 and I.3.25]. Also $\sigma_{ek}(T) \subset \sigma_{ej}(T)$ for $k < j$, the inclusion between any pair being strict in general. As mentioned, each of the sets $\sigma_{ek}(T)$ has been referred to as the essential spectrum of T : $\sigma_{e1}(T)$ is that used by Kato [25], $\sigma_{e2}(T)$ by Akhiezer and Glazman [1], and $\sigma_{e5}(T)$ by Browder [3].

Remark : By [11, Theorem I.3.7], $R(T - \lambda I)$ is closed if, and only if, $R(T^* - \bar{\lambda} I)$ is closed, and in this case,

$$\mathrm{nul}(T - \lambda I) = \mathrm{def}(T^* - \bar{\lambda} I) \quad \text{and} \quad \mathrm{def}(T - \lambda I) = \mathrm{nul}(T^* - \bar{\lambda} I).$$

A consequence of this and the fact that

$$\rho(T) = \{ \lambda \in \mathbb{C} : (T - \lambda I) \in F(H),\ \mathrm{nul}(T - \lambda I) = \mathrm{def}(T - \lambda I) = 0 \}$$

is that $\rho(T) = \overline{\rho(T^*)}$. Also, if $\sigma^*_{e2}(T) = \mathbb{C} \setminus \Phi_-(T)$, then $\lambda \in \sigma^*_{e2}(T)$ if, and only if, $\bar{\lambda} \in \sigma_{e2}(T^*)$.

Theorem 0.22 : ([11, Theorem IX.1.1])

Let T be a closed densely defined operator in H. For $k = 1,3,4,5$, $\lambda \in \sigma_{ek}(T)$ if, and only if, $\bar{\lambda} \in \sigma_{ek}(T^*)$.

Definition 0.23 : A sequence $\{ u_n \}_{n \in \mathbb{N}}$ in D(T) is called a singular sequence of T corresponding to $\lambda \in \mathbb{C}$ if it contains no convergent subsequence in H and satisfies $\| u_n \|_H = 1$ $(n \in \mathbb{N})$ and $(T - \lambda I)\, u_n \longrightarrow 0$ in H as $n \longrightarrow \infty$. This is the notion used by Akhiezer and Glazman [1] and which they call the continuous spectrum of T.

Theorem 0.24 : ([11, Theorem IX.1.3])

Let T be a closed densely defined operator in H. Then

(i) $\lambda \in \sigma_{e2}(T)$ if, and only if, there exists a singular sequence of T corresponding to λ,

(ii) $\lambda \in \sigma^{*}_{e2}(T)$ if, and only if, there exists a singular sequence of T^{*} corresponding to $\bar{\lambda}$.

Remark 0.25 :

If T is self - adjoint the sets $\sigma_{ek}(T)$, (k = 1,2,3,4,5) coincide, while, if T is J - self - adjoint $\sigma_{ek}(T)$, (k = 1,2,3,4) are identical, see [11, Theorem IX.1.6].

Definition 0.26 : Let S and T be operators acting in H with S an extension of T. We say that S is an m - dimensional extension of T if the quotient space { D(S)/D(T) } is of dimension m, i.e., there is an m - dimensional subspace G of D(S) such that

$$D(S) = D(T) \dot{+} G .$$

Theorem 0.27 : ([11, Theorem IX.4.1])

Let S be a closed m - dimensional extension of the closed, densely defined operator T in H. Then,

(i) $\text{nul}(T) \le \text{nul}(S) \le \text{nul}(T) + m$,

(ii) $\text{def}(S) \le \text{def}(T) \le \text{def}(S) + m$,

(iii) $T \in F(H)$ if, and only if, $S \in F(H)$ and

$$\text{ind}(S) = \text{ind}(T) + m .$$

Corollary 0.28 : ([11, Corollary IX.4.2])

Let S be a closed m – dimensional extension of the closed, densely defined operator T in H. Then

$$\sigma_{ek}(T) = \sigma_{ek}(S) \qquad (k = 1,2,3),$$

$$\Delta_k(T) \cap \Delta_k(S) = \emptyset \qquad (k = 4,5).$$

Thus $\sigma_{ek}(T) \ne \sigma_{ek}(S)$ for $k = 4,5$ unless

$$\sigma_{ek}(T) = \sigma_{ek}(S) = \mathbb{C} .$$

Theorem 0.29 : Let A be a closed symmetric operator with deficiency indices (m,m) and suppose that some $\lambda_o \in \mathbb{R}$ is such that $\lambda_o \notin \Pi(A)$. Then if S is any closed symmetric extension of A, either λ_o is an eigenvalue of A and S or λ_o is not an eigenvalue of A and range $R(A - \lambda_o I)$ and range $R(S - \lambda_o I)$ are both not closed (in the latter case $\lambda_o \in \sigma_{e1}(S)$). Thus $\mathbb{R} \setminus \Pi(A) \subset \sigma(S)$ for every symmetric (or indeed finite dimensional extension of A); see Theorem 0.27.

In particular the above Theorem implies that if A has no eigenvalues, then $\mathbb{R} \setminus \Pi(A)$ lies in the essential spectrum $\sigma_{e1}(S)$ of every closed symmetric extension S (in fact of every closed finite dimensional extension of A).

Remark 0.30 Let A be a closed symmetric operator and

$$\text{nul}(A^* - \lambda_0 I) = k, \qquad (\lambda_0 \in \mathbb{R}),$$

$$\text{def}(A - \lambda I) = m, \qquad (\text{Im}\, \lambda \neq 0)$$

with $k < m$. Then $\lambda_0 \notin \Pi(A)$.

Definition 0.31 Given any normed vector space X, by the adjoint space X^* of X is meant the set of all conjugate linear continuous functionals on X; that is, $f \in X^*$ if, and only if, $f : X \longrightarrow \mathbb{C}$ is continuous and

$$f(\alpha x + \beta y) = \bar{\alpha}\, f(x) + \bar{\beta}\, f(y),$$

for all $\alpha, \beta \in \mathbb{C}$ and all $x, y \in X$.

Definition 0.32 : Let X be a normed vector space and let $G \subset X$. The annihilator $G^{\perp}$ of G in X^* is defined by

$$G^{\perp} = \{ u \in X^* : \langle u,g \rangle = 0 \text{ for all } g \in G \} ;$$

if $F \subset X^*$, then $F^{\perp}$, the annihilator of F in X, is defined by

$$F^{\perp} = \{ v \in X : \langle f,v \rangle = 0 \text{ for all } f \in F \} ,$$

where $\langle f,v \rangle$ denotes $f(v)$.

Remark : If X is an inner - product space with inner product $(.,.)$, we say that u is orthogonal to a subset G of X in symble $u \perp G$, if

$$(u,g) = 0 \qquad \text{for all } g \in G.$$

Notation : Let I be a real interval with end points a,b ($-\infty \leq a < b \leq \infty$). By $L_{loc}(I)$ we denote the set of functions on I which are Lebesgue integrable over every compact subinterval of I. Let B(I) denote the set of measurable complex - valued functions on I. We denote by AC(I) the set of functions in B(I) which are absolutely continuous on the interval I and by $AC_{loc}(I)$ the set of functions in B(I) which are absolutely continuous on every compact subinterval of I.

In this thesis, Chapter II is divided into two sections and Chapter III is divided into three sections, and each section into subsections. For example, § II.1.1 means subsection 1 of section 1 in Chapter II. It is written § 1.1 inside the same chapter. Theorems, Corollaries, Lemmas and Remarks are numbered consecutively within each section. Definition I.1.1 means Definition 1.1 in Chapter I, and Theorem III.3.10 means Theorem 3.10 in Chapter III and is referred to as Theorem 3.10 within the same chapter. Formulas are numbered consecutively inside each section, e.g. (III.3.17) means the seventeenth equation in Chapter III and is referred to as (3.17) within the same chapter.

There is a list of references at the end of this thesis, which have been consulted or which are of great importance to the problems considered in the thesis.