ABSTRACT

H. Weyl [47] classified the differential equation

$$f''(x) + (\lambda - q(x))f(x) = 0 \quad , \quad [\, 0 \le x < \infty \,] \qquad (1)$$

when $q(x)$ is real-valued as being in the limit-circle case (LC) at ∞ if all solutions of (1) are square integrable, i..e.,

$$\int_0^\infty |f(x)|^2\,dx < \infty \;;$$

otherwise. the equation (1) is said to be in the limit-point case (LP) at ∞. This classification does not depend on the complex parameter $\lambda=\mu+I\nu$.

Sims [40] proves that a restricted form of Weyl's limit-point, limit-circle theory for the equation (1) holds if $q(x)$ is complex-valued, $\text{Im}(\lambda)$ and $\text{Im}(q(x))$ are of opposite sign for all x. Sims' result is that there is at least one solution $f(x)$ of (1) which is such that

$$\int_0^\infty |\text{Im}(\lambda - q(x))|\,|f(x)|^2\,dx < \infty.$$

According to Sims theory, there are three possible cases of classification of equation (1) as follows:

Case I :There is precisely one and only one solution which belongs to $L^2(a,b;(\gamma - q_2))$. In addition it is the only solution in $L^2(a,b)$

Case II:There is precisely one and only one solution which belongs to $L^2(a,b;(\gamma - q_2))$but all solution belong to $L^2(a,b)$

Case III:All solution are in $L^2(a,b) \cap L^2(a,b;(\gamma - q_2))$.

Abstract

Here we generalize the results of H. Weyl and Sims for the general second-order quasi-differential equation in the form

$$-(p(f'-rf))' + up(f'-rf) + qf = \lambda wf \qquad \text{on } [a,b) \qquad (2)$$

and its formal adjoint

$$-(\bar{p}(g'+\bar{u}g))' - \bar{r}\bar{p}(g'+\bar{u}g) + \bar{q}g = \lambda wg \qquad \text{on } [a,b) \qquad (3)$$

where the coefficients q, r and u are complex-valued and p, w are positive real-valued functions Lebesgue measurable on the interval $[a,b)$ of the real line R and satisfy some basic conditions; $\lambda=\mu+i\nu$ is a complex parameter .

Thus, according to the definitions in [11], [20], [33] and [47] the quasi-differential expression

$$M[f] = -(p(f'-rf))' + up(f'-rf) + qf \qquad \text{on } [a,b) \qquad (4)$$

is regular at the end-point a and is singular at the end-point b.

The thesis contains of five chapters:

Chapter I contains definitions and facts of the theory of linear operators pertinent to later chapters.

Chapter II is concerned with the relation between the concepts of the deficiency indices of the linear operators $T_o(M)$ and $T_o(M^+)$ generated by M and M^+, which satisfy the following inequality

$$2 \le \operatorname{def}[T_o(M) - \lambda I] + \operatorname{def}[T_o(M^+) - \bar{\lambda} I] \le 4 \quad \text{for all } \lambda \in \Pi(T_o(M), T_o(M^+)).$$

In **Chapter III**, we study the number of $L^2_w(a,b)$ solutions of (2) with $u=-\bar{r}, M=M^+$. The coefficients p and q are real-valued and r is a complex-valued functions defined on $[a,b)$. In view of the symmetricty of $M[.]$

it follows from the general theory in [33; see Section 17.5], [44] and the original definition of H. Weyl [47] that the differential expression $M[.]$ may be classified as either Limit-point (LP) or limit-circle (LC) at the singular end-point b according to whether the differential equation $M[f]= \lambda wf$ has, respectively, exactly one or two linearly independent solutions in the weighted Hilbert space $L^2_w(a,b)$ when $Im(\lambda)\neq 0$. This classification depends only on the nature of the coefficients p, q, r and w. There are no necessary and sufficient conditions on p, q, r and w to distinguish between limit-point and limit-circle cases at the singular end-point b, but there is a necessary and sufficient condition in terms of certain functions in the weighted Hilbert space $L^2_w(a,b)$.

In **Chapter IV**, we proved the Sims theory of (2) with $u=-\bar{r}$ and $M^+=\overline{M}$ with the coefficient q is a complex-valued function defined on $[a,b)$ and the three cases above.

In **Chapter V** we consider the case $M\neq M^+$ and we obtain the results of Chapter III as a special case of the results in Chapter V.