Chapter(V)

Results

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The main objective of the present thesis is devoted to the approximate solution of two boundary value problems. The first boundary value problem is concerned with the steady state rotational flow, where the z-axis is taken as the axis of rotation, about a sphere of radius "R" in an infinite domain of a viscoelastic fluid. A simultaneous rotational and translational motions of the sphere in the same viscoelastic fluid is the subject of the second boundary value problem. In the two cases the fluid is modeled by the general Oldroyd constitutive equation containing of six viscometric parameters; namely η_0, λ_1, λ_2, λ_5, λ_6 and λ_7. The elasticity of the fluid is assumed to dominate the inertia such that the later can be neglected in the momentum equation. An approximate solution is obtained through the expansion of the dynamical variables in a power series of the retardation parameter λ. Hence, a set of successive partial differential equations is obtained . The solution of the first boundary value problem is performed up to the third-order approximation, while the solution is performed up to the second-order approximation for the second boundary value problem .

For simplicity, we shall consider the final results of the two boundary-value problems separately.

(i) The first boundary value problem (rotational motion only) includes these results:

In the zero-order approximation the primary velocity field $W^{(0)}(r,\theta)$ has only a φ-component and it is proportional to Ω. This is called the Newtonian velocity. Moreover, there is no secondary flow in this order of approximation.

For the first-order approximation, the stream function $\Psi^{(1)}(r,\theta)$ represents the secondary flow field in the ρz-plane and it is proportional to $(\lambda_1 - \lambda_2)$. This secondary flow describes an Oldroyd-B fluid flow and it results from the normal stresses due to the elasticity of the fluid, which is being towards the sphere near the equator and away from it near the axis of rotation, with two zones separated by conical surfaces given by $3\cos^2\theta - 1 = 0$, or $\theta = \arccos(1/\sqrt{3}) = 54^\circ 44'$ or $125^\circ 16'$. This behavior is in qualitative agreement with the flow pattern observed experimentally [13]. Moreover this secondary flow field $\Psi^{(1)}(r,\theta)$ is in contrast to the flow due to the inertia which is being away from the sphere near the equator and towards to it near the axis of rotation. Finally, this flow field has no effect on the torque acting on the sphere since the torque results from the stress component $T_{r\varphi}$ which depends only the φ-component velocity $W^{(n)}(r,\theta)$ with n=0,2.

The second-order approximation leads to the viscoelastic contribution $W^{(2)}(r,\theta)$ superimposed upon the leading term $W^{(0)}(r,\theta)$, which enhances or opposes the $W^{(0)}(r,\theta)$ flow field. This second-order term depends on all the viscometric parameters η_0, $\lambda_1, \lambda_2, \lambda_5, \lambda_6$ and λ_7 of the fluid under consideration.

The third-order stream-function $\Psi^{(3)}(r,\theta)$ depends also on all of the aforementioned viscometric parameters. The normal stresses are more effective in this order of approximation where they try to recede the fluid from the sphere in the direction of the axis of rotation and towards the equator of the sphere in accordance with Weissenberg climbing effect [1]. To the best of our knowledge this is being the first analytical solution which satisfies this effect up to the third-order. The secondary flow field $\Psi^{(3)}(r,\theta)$=const. is shown on Figs.(4a,...c) for different types of viscoelastic model fluids; namely, Maxwell , Oldroyd-B and the general Oldroyd model fluids respectively.

The calculated torque acting on the sphere is composed of a viscoelastic contribution $\tilde{M}_2$ in addition to the viscous term $\tilde{M}_\circ$. The present results show that this torque is given by,

$$\tilde{M} = 8\pi\Omega R^3\eta_\circ[1 - \Omega^2\{\tfrac{2}{15}(\lambda_1 - \lambda_2)^2 + \tfrac{12}{5}\beta\}]$$
$$= \tilde{M}_\circ + \tilde{M}_2 \qquad (5\text{-}1)$$
$$= \tilde{M}_\circ[1 - \Omega^2\{\tfrac{2}{15}(\lambda_1 - \lambda_2)^2 + \tfrac{12}{5}\beta\}]$$

where $\tilde{M}_\circ = 8\pi\Omega R^3\eta_\circ$ and $\tilde{M}_2 = 8\pi\Omega^3 R^3\eta_\circ[\{\tfrac{2}{15}(\lambda_1 - \lambda_2)^2 + \tfrac{12}{5}\beta\}]$

An experimental curve of $(\tilde{M}_\circ - \tilde{M})/\tilde{M}_\circ$ against Ω^2 can be used to estimate the value of parameter $[\tfrac{2}{15}(\lambda_1 - \lambda_2)^2 + \tfrac{12}{5}\beta]$, [15].

This viscoelastic contribution may enhance or oppose the viscous term depending on the values of the material parameters of the fluid under consideration . However, if Oldroyd-B fluid is considered, the viscoelastic contribution adds negatively to $\tilde{M}_\circ$. This result coincides with the obtained results by Thomas et al.[15] for a sphere rotating in an Oldroyd-B fluid including the inertial term. Their torque result is given as :

$$M = 8\pi\eta_\circ\Omega a^3\left[1 + L(\tfrac{1}{1200} + \tfrac{m}{140} - \tfrac{2m^2}{15})\right] \qquad (5\text{-}2)$$

where ,

$m = \eta_\circ(\lambda_1 - \lambda_2)/\rho a^2$, $L = (\frac{\Omega a^2}{\nu})^2$ and $\nu = \frac{\eta_\circ}{\rho}$

The first two terms in Eq.(5-2) result from the inertial term, and the third term is an effect of elasticity. A comparison with the present work reveals a good agreement for the elasticity term in the last Eq.(5-2) in spite of the different of the expansion parameters i.e λ and L .

Moreover, using the retarded motion expansion method for third order fluid including inertia, Walters et al.[12,14] showed that the torque acting on a rotating sphere about its diameter is given by :

$$M = 8\pi b_1 R^3 \Omega \left[1 + \frac{Re^2}{1200} + \frac{Re\,De}{140}(1 + \frac{b_{11}}{b_2}) - \frac{2De^2}{15}(1 + \frac{b_{11}}{b_2})^2 - \frac{24De^2}{5}\frac{b_1(b_{12} - b_{1:11})}{b_2^2} \right] \quad (5\text{-}3)$$

where $Re = R^2\Omega\rho / b_1$ and $De = -(b_2 / b_1)\Omega$ are the Reynolds and Deborah numbers, which represent the viscosity and the elasticity forces; respectively. The last two terms in Eq.(5-3) are due to elasticity. A comparison of these two terms gives a satisfactory agreement results with the present work. Therefore, we can distinguish a relation between the constants for the two models as follows; table(5-1) .

Table(5-1). Retarded motion expansion constants and general Oldroyd model

Order	Retarded motion expansion	General Oldroyd model
1	b_1	$\eta_\circ$
2	b_2 b_{11}	$-\eta_\circ(\lambda_1 - \lambda_2)$ 0
3	b_3 b_{12} $b_{1:11}$	$\eta_\circ\lambda_1(\lambda_1 - \lambda_2)$ 0 $1/2\eta_\circ[(\lambda_7 - \lambda_6)(\lambda_1 - 3/2\lambda_5) - (\lambda_1 - \lambda_2)\lambda_5]$

(ii)The second boundary value problem (rotational and translational motions) includes the following results:

The zero-order velocity $W^{(0)}(r,\theta)$ as well as the zero-order stream function $\Psi^{(0)}(r,\theta)$ are the same as in the corresponding Newtonian cases.

Up to the first-order approximation the velocity $W^{(1)}(r,\theta)$ depends on the viscometric parameters $(\lambda_1 - \lambda_2)$. Due to its symmetrical behavior it doesn't affect the torque acting on the sphere; Fig(8a).On the other hand, the first-order stream-function $\Psi^{(1)}(r,\theta)$ depends also on the same viscometric parameters $(\lambda_1 - \lambda_2)$ and due to its symmetrical behavior it doesn't affect on the drag on the sphere; Fig(8c).

For the second-order approximation, the second-order velocity $W^{(2)}(r,\theta)$ superimposed onto the flow field $W^{(0)}(r,\theta)$, enhancing or diminishing it depending on the values of the six material parameters $\eta_0, \lambda_1, \lambda_2, \lambda_5, \lambda_6$ and λ_7. As a special case, for Oldroyd-B fluid $W^{(2)}(r,\theta)$ field diminishes the primary flow $W^{(0)}(r,\theta)$, so that the torque acting on the sphere decreases in comparison with its value in Newtonian flow according to the formula following:

$$\begin{aligned}\widetilde{M} &= 8\pi\Omega R^3\eta_\circ - \frac{2\pi\Omega^5 R^5 \eta_\circ}{V_\circ^2}\left[(\lambda_1 - \lambda_2)(2.2\lambda_1 + 2.0735\lambda_2) + 12.49\beta^\circ\right] \\ &= \widetilde{M}_\circ + \widetilde{M}_2\end{aligned}$$

(5-4)

where ,

$\tilde{M}_\circ = 8\pi\Omega R^3 \eta_\circ$, is the viscous torque

and,

$$\tilde{M}_2 = -\frac{2\pi\Omega^5 R^5 \eta_\circ}{V_\circ^2}\left[(\lambda_1 - \lambda_2)(2.2\lambda_1 + 2.0735\lambda_2) + 12.49\beta^\circ\right]$$

is the viscoelastic contribution.

An experimental curve of $(\tilde{M}_\circ - \tilde{M})/\tilde{M}_\circ$ against $\frac{\Omega^4 R^2}{V_\circ^2}$ could in fact be used to estimate the value of the viscometric parameter $\frac{1}{4}[(\lambda_1 - \lambda_2)(2.2\lambda_1 + 2.0735\lambda_2) - 12.49\beta^\circ]$,[15] .

As a conclusion for the general Oldroyd fluids, the torque is decreasing or increasing in accordance with the values of the viscometric parameters of the fluid under consideration.

Moreover, the second-order stream-function $\Psi^{(2)}(r,\theta)$ depends on all the viscometric parameters η_0, λ_1, λ_2, λ_5, λ_6 and λ_7 of the fluid. The present analytical solution for $\Psi^{(2)}(r,\theta)$ enables us to study the effect of the normal stresses associated with an extra tension along the stream-lines; [8] . As a special case, for Oldroyd-B fluid the resulting normal stresses push the streamlines away from the sphere so it decreases the drag on the sphere as follows:

$$\widetilde{D} = 6\pi R\eta_\circ V_\circ - \frac{2\pi R^3\Omega^4\eta_\circ}{V_\circ}[(\lambda_1 - \lambda_2)(4.73\lambda_1 + 5.865\lambda_2) + 5.293\beta^\circ]$$
$$= \widetilde{D}_\circ + \widetilde{D}_2 \tag{5-5}$$

where,

$\widetilde{D}_\circ = 6\pi R\eta_\circ V_\circ$, is the viscous contribution

and,

$$\widetilde{D}_2 = -\frac{2\pi R^3\Omega^4\eta_\circ}{V_\circ}[(\lambda_1 - \lambda_2)(4.73\lambda_1 + 5.865\lambda_2) + 5.293\beta^\circ],$$

is the viscoelastic contribution.

An experimental curve of $(\widetilde{D}_\circ - \widetilde{D})/\widetilde{D}_\circ$ against $\frac{\Omega^4 R^2}{V_\circ^2}$ could in fact be used to estimate the value of the parameter $\frac{1}{3}[(\lambda_1 - \lambda_2)(4.73\lambda_1 + 5.865\lambda_2) + 5.293\beta^\circ]$, [8].

Hence, for the general Oldroyd fluid the resulting normal stresses push the streamlines towards the sphere and this increases the drag on the sphere for $\beta = -0.5$ and -1.

Finally, using the slopes gained from Eqs.(5-1,4,5) the values of the material parameters $\lambda_1, \lambda_2, \beta^\circ$ can be determined exactly.

On the basis of this theoretical calculation, a rheometer can be designed in order to determine the viscometric parameters η_0, λ_1, λ_2, λ_5, λ_6 and λ_7.