Preface

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The notion of a fuzzy set, introduced by Lotfi A. Zadeh ([65]) in 1965, has caused great interest among both pure and applied mathematicians. It has also raised enthusiasm among some engineers, biologists, phychologists, economists, and experts in other areas, who use (or at least try to use) mathematical ideas and methods in their research. By this notion of fuzzy set it is possible to obtain a more distinctive description of some phenomena than the one which is offered by systems based on classical two-valued logic and classical set theory. In the fuzzy set theory there appear linguistic expressions such as "a person is tall". In the crisp point of view one may say that a person is tall if his height is greater than 185 cm whereas in the fuzzy point of view this property can be determined by a function on a scale of centimetres with values in the closed real unit interval $[0,1]$. There are many realized technical applications showing the practical usefulness of fuzzy theory. The first applications of fuzzy theory appear in the late seventies. They were primarily industrial such as process control for cement kilns. In 1987 in Sendai, Japan a fuzzy controlled subway system started operation. Beginning with that time, fuzzy control is more and more technically applied. In particular, a lot of consumer products, made in Japan, is based on fuzzy technology. Examples are the following: fuzzy washing machine, fuzzy air conditioner and fuzzy vacuum cleaner. Moreover, fuzzy control is used in camcorders, television sets, cameras (eliminating jumping of pictures), cars (electronic fuel injection controls, automatic cruise control system), and so on.

General topology was one of the first branches of pure mathematics to which fuzzy sets have been applied. It was in 1968, that is, three years after Zadeh's paper had appeared, that Chang ([6]) introduced the notion of fuzzy topology. Several other authors continued the investigation of such spaces such as Eklund and Gähler ([11]), Gähler ([21, 24]), Geping and Lanfang ([31]), Goguen ([33]), Katsaras and Petalas ([42]), Kerre and Ottoy ([44]) and Lowen ([47, 48, 49, 50, 51, 52, 53]). In Goguen's paper ([33]) he considered as values of the fuzzy sets not only elements of the closed unit interval but elements of some more general lattice L. The notion of stratified fuzzy topology has been introduced by Lowen ([51]). By a fuzzy topology to be stratified is meant that all constant fuzzy sets are open. Lowen defined also the notion of induced fuzzy topology of an ordinary topology as a special stratified fuzzy topology. Acceptable notions of fuzzy proximity were given by Artico and Moresco ([2, 3]) and by Katsaras ([40]). Some definitions of fuzzy uniform structure neighborhood structure.

Note that R. Lowen defined in [52] a notion of fuzzy neighborhood space using only prefilters. As a consequence, fuzzy neighborhood spaces in his sense are, up to identifications, only special fuzzy topological spaces (cf. the characterizations in [56]).

The thesis consists of three chapters and is organized as follows:

Each chapter begins with an introduction, in which are given some informations, ideas and motivations on the contents of this chapter.

Chapter 1 is the basic chapter of the thesis which contains the definition of the fuzzy neighborhood structure and its modification obtained by restricting to homogeneous fuzzy filters. In this chapter it is shown that, up to an identification, the category $\mathsf{FNS_h}$ of homogeneous fuzzy neighborhood spaces is a full and bicoreflective subcategory of the category FNS of fuzzy neighborhood spaces. It is also shown that the full category FTOP of fuzzy topological spaces is a full and bicoreflective subcategory of FNS and that the subcategory of FTOP of all stratified fuzzy topological spaces is a full and bicoreflective subcategory of $\mathsf{FNS_h}$. In **Section 1.2**, we recall some of the definitions and results about fuzzy filters needing throughout the thesis. **Section 1.3** deals with the description of the first type of the fuzzy neighborhood structure. The case of fuzzy topology is considered and it is shown that any fuzzy topology is, up to an identification, a special fuzzy neighborhood structure of the first type. **Section 1.4** is devoted to the homogeneous the fuzzy neighborhood structure. The fuzzy topology associated to a homogeneous fuzzy neighborhood structure is defined analogously as the fuzzy topology associated to a fuzzy neighborhood structure in the general sense. In this section also the stratified fuzzy neighborhood structures are defined. Moreover, the relations between the first type and the second type of fuzzy neighborhood structures are investigated. In case of a fuzzy topology, considered as a special fuzzy neighborhood structure, stratified in sense of fuzzy neighborhood structure coincides with the property of this fuzzy topology to be stratified in the usual sense. In **Section 1.5**, it is shown that for any isotone mapping $h : \mathcal{F}_L X \to \mathcal{F}_L X$ which has all properties of a fuzzy neighborhood structure may be with except of (N2) by a "basic construction" the greatest fuzzy neighborhood structure less than or equal to h is constructed. Applying this basic construction, the stratification of a fuzzy neighborhood structure h, that is, the coarsest stratified fuzzy neighborhood structure finer than h, is introduced. In this section also it is shown that the full subcategory of FNS of all stratified fuzzy neighborhood spaces is a bicoreflective subcategory of FNS. In **Section 1.6** we show, using the "basic construction", that in the category FNS all initial and final lifts exists, that is, FNS is a topological category. Hence FNS is complete and cocomplete. As a consequence, for instance, all products and sums of fuzzy neighborhood spaces exist.

Noting that some of the results in Chapter 1 are submitted for publication in [27] and the most of results in this chapter are accepted for publication in [28].

In **Chapter 2** we study fuzzy topogenous orders, in particular, fuzzy topogenous structures and the more special fuzzy proximities in sense Katsaras ([40]). In this chapter also a modification of the notion of fuzzy proximity is considered, called *fuzzy proximity of the internal type*. In **Section 2.2** we show that fuzzy topogenous structures are, up to an identification, special fuzzy neighborhood structures. More generally, we show that fuzzy topogenous orders can be characterized by the notion of fuzzy neighborhood prestructure which is a weakening of that of fuzzy neighborhood structure. It is shown that up to our identifications the category FTS of fuzzy topogenous spaces is a full subcategory of the category FNS of fuzzy neighborhood spaces. In **Sections 2.3** and **2.4**, the notions of perfect and biperfect fuzzy topogenous orders are studied. Moreover, fuzzy relations are considered and it is shown that special and important type of biperfect fuzzy topogenous orders can be characterized by fuzzy relations. In **Section 2.5**, stratified fuzzy topogenous orders and structures are defined. It is shown that for each stratified fuzzy topogenous structure the characterizing fuzzy neighborhood structure is also stratified. Here also for fuzzy topogenous structure the related notion of stratification is introduced and some results related to the stratification of the characterizing fuzzy neighborhood structures are obtained. **Section 2.6** is devoted to initial and final fuzzy topogenous orders. In **Section 2.7**, we study fuzzy proximities. The property of symmetry of a fuzzy proximity depends on a fixed order-reversing involution of the given lattice L. In this section a notion of symmetry of fuzzy neighborhood structure is introduced which is defined independently on a fixed order-reversing involution of L. In specializing to fuzzy topogenous structures for these structures a new notion of symmetry is introduced called *internal symmetry*. The internally symmetric fuzzy topogenous structures give a new notion of fuzzy proximity, called fuzzy proximity of the internal type. It is shown that a fuzzy topogenous structure is a fuzzy proximity of the internal type if and only if the associated fuzzy neighborhood structure is symmetric. In **Section 2.8**, the notion of stratified fuzzy proximity and the related notion of stratification are considered. Using this stratification of fuzzy proximities usual proximities can be embedded canonically into the fuzzy case.

Let us here mention that some of the results in Chapter 2 are submitted for publication in [27] and the most of results in this chapter are submitted for publication in [29].

Chapter 3 is devoted to a new notion of fuzzy uniform structure. In **Section 3.2**, we study special fuzzy filters, called *relational fuzzy filters* and *sup-fuzzy filters*. Examples of sup-fuzzy filters are the homogeneous fuzzy filters and the fuzzy filters which only have 0 and 1 as values, called 0, 1-fuzzy filters. In **Section 3.3**, by means of the relational fuzzy filters, sup-fuzzy filters. 0, 1-fuzzy filters and homogeneous fuzzy filters we define the fuzzy uniform structures, sup-fuzzy uniform structures, $0,1$-fuzzy uniform structures and homogeneous fuzzy uniform structures, respectively. Fuzzy uniform structures in sense of Katsaras ([38]) are, up to identification, $0,1$-fuzzy uniform structures. In **Section 3.4**, it is shown that the sup-fuzzy uniform structure can be characterized by a related notion of fuzzy sytopogenous structure. The $0,1$-fuzzy uniform structures coincide with the fuzzy syntopogenous structures proposed by Katsaras and Petalas in [42] in a special case. Here we compare also our notion of fuzzy uniform structure with that of Hutton ([35]). In **Section 3.5** we show that the fuzzy metrics in sense of [18] generate canonically homogeneous fuzzy uniform structures in our sense. In **Section 3.6**, to each fuzzy uniform structure $\mathcal{U}$ is associated a homogeneous fuzzy neighborhood structure h. In case of a $0,1$-fuzzy uniform structure and the more general case of a cascade fuzzy uniform structure even a fuzzy neighborhood structure of the first type can be associated, which is stratified. For the fuzzy uniform structure in sense of Katsaras, h is, up to an identification, a fuzzy proximity of the internal type.

Noting that the most of results in Chapter 3 are submitted for publication in [30].