INTRODUCTION

The thrust of this thesis is concerned with the crossed product R * G of a group G mainly polycyclic-by-finite group or more specified an infinite dihedral group over a ring R.

The importance of this topic is not just a meeting place for group theory and ring theory but because it is a straight generalization of many important algebraic structures such as group rings, skew group rings and twisted group rings.

Recently, there has been a growing ring theoretic interest in crossed products and in particular in skew group rings of finite groups this is due mainly to their relationship to a possible Galois theory for rings. For instance, there are strong ties between skew group ring RG and a fixed Galois subring R^G. Indeed, for the most of the decent structures those two rings are Morita equivalent. These topics will be discussed in more details.

Moreover, crossed product is closely related to the study of finite dimensional division algebras, central simple algebras and their Brauer groups, group graded rings.

The theory of central simple algebras rests on the fact that every such algebra is similar to a crossed product algebra over the same field or commutative ring K. On other words these two algebras coincide (belong to the same class) in the Brauer group B(K) whose elements are classes of central simple algebras containing the same maximal subfield. The fact that the crossed product of a field with its Galois group is central simple algebra raises an important question: when is a central simple algebra is a crossed product? (and not just similar to it). It is well known that this question is equivalent to when is the maximal subfield of this central simple algebra is a normal extension of the center K?. For instance if the algebra is a cyclic algebra then this question has an affirmative answer.

Moreover it was shown that central division algebras of degree 2,3,4,6,12 are CROSSED product. However in 1972 S. Amitsur constructed examples of finite dimensional central division algebras which are not CROSSED products.

There are many reasons to focus in this thesis on the so-called algebras of type E, i.e the infinite cyclic or infinite dihedral crossed product: one of these reasons is that any infinite supersolvable group has a homomorphic image which is one of above two groups. The other reason and perhaps the effective one in this thesis is that the structure of finitely generated modules over the group algebra KG of a group G containing an infinite cyclic group of finite index can be reduced to the study of Matrices over algebras of type E.

This thesis consists of six chapters.

Chapter 0: Contains most of the basic definition and terminology used throughout.

Chapter I: Studies the crossed products from different directions and show their relations with central simple algebras and their Brauer groups.

Chapter II: Gives a brief demonstration on the previous results and that any group G containing an infinite cyclic subgroup of finite index (or infinite supersolvable group) can be transfered into either infinite cyclic group or an infinite dihedral group. Also we give the structural theorem of finitely generated KG modules for such groups G and show that they equal the direct sum of matrices over algebras of type E. These results are mainly due to Berman and Buzási.

Chapter III: Gives in details all types of algebras E over a finite field. It is shown that there is exactly (up to isomorphism) eight types which we denote by A_1 - A_8.

Chapter IV: Studies the Realization of algebras of type E. More precisely we show that for each algebra A_i there is a group G_i such that A_i is isomorphic to some (principal) ideal in the group algebra KG_i.

Chapter V: Consists of eleven sections each section deals with one or more important algebraic property and examine its existance in each type A_i.

For instance in section 5.1 we show that algebras of type E (even over Noetherian rings) are Noetherian and Jacobson algebras.

In section 5.2 we show that algebras of type E are prime and semiprimitive. Also, we show that every ideal in such algebra is fractional.

In section 5.3 we show that algebras of type E satisfy a polynomial identity and hence we conclude a lot of other algebraic properties satisfied by such algebras. One of these properties that algebras of type E are fully bounded (FBN).

In sections 5.4 & 5.5 we study the dimensions of algebras of type E. We prove that the classical Krull dimension, nonclassical Krull dimension & Global dimension are all equal one, and we deduce that algebras of type E are

Hereditary but neither Artinian nor von-Neumann regular.

Section 5.7 studies primitivity & ;simplicity of algebras of type E we conclude using different techniques that Algebras A_2, A_7 and A_8 are simple (=> primitive) while that rest algebras of type E are not.

In Section 5.8 we consider the zerodivisibility locality and Ore conditions for algebras of type E. We show that all algebras of type E, except A_1 & A_2 contains zero-divisors. We deduce that algebras A_3, A_4, A_6, A_7 & A_8 are not local. We show also that algebras A_1, A_2, A_3 and A_7 are Ore rings.

Section 5.9 studies the principality of ideals of algebras of type E. Although the study of this important property was one of motivations of this thesis, we believe that the results of this section are complete. However, we were able to show that algebras A_1, A_2 & A_5 with $b^2 = -1$ are principal ideal rings (The last case is due to P. Pálfy), while algebra A_3 is not.

In section 5.10 we investigate a more general and more related property to the property of principal ideal ring and namely: Unique factorisation domain (UFD) and we show that algebra A_3 is not a UFD, which emphasizes the results of section 5.9.

Finally in section 5.11 we study the Grothendieck group for algebras of type E. We show that the Greothendieck group of $A_5 \ldots A_8 \approx Z^3$ & the Grothendieck group for $A_1 \approx Z$

The results of chapters III & IV are published in [2] & [3], while the results of V are under preparation [4].