CHAPTER 0

DEFINITIONS TERMINOLOGY AND BASIC RESULTS

§ 0.1 Modules

Simple, Indecomposable, Faithful

Definition 0.1.1. A right R-module M is called simple (or Irreducible) iff i) $MR \neq 0$ ii) M has no nonzero proper R-submodules.

Definition 0.1.2. A right R-module M is called decomposable iff $M = \Sigma^{\oplus} M_i$ where M_i are nonzero proper R-submodules. Otherwise M is called indecomposable.

Definition 0.1.3. A submodule N_R of M_R is called an essential submodule iff $N_R \cap M^1{}_R \neq \{0\}$ for every nonzero submodule M^1 of M in this case.

Definition 0.1.4. A module M_R is called faithful if it has a zero annihilator, i.e. $Ann_R(M) = \{x \in R \mid Mx = 0\} = \{0\}$

Definition 0.1.5. A module P_R is called projective iff one of the following conditions holds

i) For every epimorphism $\pi: B_R \rightarrow A_R$ and homomorphism $\Phi: P_R \rightarrow A_R$, then there is a homomorphism $\Psi: P_R \rightarrow B_R$ such that $\pi o \Psi = \Phi$.

ii) Every epimorphism $\pi: B_R \rightarrow P_R$ is direct i.e. there exists a monomorphism $\theta: P \rightarrow B$ such that $\pi o \theta =$ identity homomorphism.

iii) If P_R is a homomorphic image of a module B_R then P_R is a direct summand of B.

Definition 0.1.6. A module M_R is called flat iff whenever $K: A_R \rightarrow B_R$ is mono, then $K \otimes 1 : A \otimes_R M \rightarrow B \otimes_R M$ is also monomorphism.

Definition 0.1.7. A module I_R is called injective iff one of the following conditions holds

i) $\forall$ monomorphism $K:A_R \to B_R$ and homomorphism $\Phi:A \to I$, then there is a homomorphism $\Psi:B_R \to I_R$ such that $\Psi o K = \Phi$.

ii) Every monomorphism $K:I_R \to B_R$ is direct i.e. there is an epimorphism $\pi:B_R \to I_R$ such that $\pi o K$ = identity

iii) if I_R is imbedded in B_R then I_R is a direct summand of B_R

iv) I_R is a direct summand of a character module of a free module.

§ 0.2 Simple Rings, Central Simple Algebras, Primitive Rings, Prime Ring and Semi Prime Rings.

Definition 0.2.1. A ring R is called simple if it is simple as R-module i.e. $R^2 \neq 0$, and R has no proper nonzero ideals.

Definition 0.2.2. An algebra A_F (over a field F) is called central if the center of A coincides with the field F.

Definition 0.2.3. A ring R is called right primitive if it has a zero right primitive ideal $I = (\rho:R) = \{x \in R \mid Rx \subseteq \rho\}$, where ρ is a maximal right ideal in R. This is equivalent to R has simple irreducible ($MR \neq 0$, M has no R-submodule) faithful ($Ann_R(M) = \{x \in R \mid Mx = 0\} = 0$) R-module M.

It is clear that if R is a simple ring with 1 then R is primitive.

Definition 0.2.4. An ideal P is called prime if for any two ideals A, B ΔR such that $AB \subset P$ implies $A \subset P$ or $B \subset P$.

Definition 0.2.5. A ring is called prime if it has a zero prime ideal P. This is equivalent to $aRb = 0$ iff $a = 0$ or $b = 0$ where $a,b \in R$.

Any primitive ideal (ring) is prime one.

Definition 0.2.6. $\cap$ all right primitive ideals $= \cap$ all left primitive ideals $= \cap$ all maximal right ideals $= \cap$ all maximal left ideals $= \cap$ Annihilators of simple right modules $= \cap$ Annihilators of simple left R-modules is called Jacobson radical of R and denoted by J(R).

Definition 0.2.7. $\cap$ all prime ideals of R is called the prime (Baer) radical of R and is denoted by P(R). P(R) is a nil ideal

Definition 0.2.8. A ring is called semi primitive iff $J(R) = 0$ and semiprime if $P(R)=0$

R is semiprime iff R has no nonzero (left, right or two-sided) nilpotent ideal.

§ 0.3 Chain Conditions

Definition 0.3.1. A module M is said to be Noetherian iff every ascending chain of submodules M_i of M is ultimately constant i.e, if there is a positive number n such that $M_n = M_i$ for all $i \geq n$.

Proposition 0.3.2. Given a module M, the following conditions are equivalent: i) M is Noetherian ii) All submodules of M are finitely generated iii) All finitely generated submodules of M are Noetherian.

Definition 0.3.3. A module M is said to be Artinian iff every descending chain of submodules M_i of M is ultimately constant

Definition 0.3.4. The ring R is right (left) Noetherian (Artinian) if $R_R({}_RR)$ is Noetherian (Artinian)

Definition 0.3.5. The left annihilator of a subset S of a ring R is the set: $Ann_\ell(S)=\{r\epsilon R \mid rS = 0\}$. Similarly we define $Ann_r(S)$.

It is evident that $Ann_\ell(S)$ and $Ann_r(S)$ are left and right ideals respectively.

Definition 0.3.6. A ring R is said to satisfy Acc (Ann_ℓ) if every ascending chain of left annihilators of subsets of R terminates

Definition 0.3.7. A ring R is said to satisfy $Acc_\ell\oplus$ if R does not contain infinite direct sum of left ideals of R

Definition 0.3.8. A ring R is left Goldie if R satisfies $Acc_\ell(Ann)$ and $Acc_\ell\oplus$.

Any Noetherian ring is a Goldie ring. The ring E $[x_1, x_2, \ldots]$ of polynomials in several indeterminates over any field is Goldie but not Noetherian

§ 0.4 Asano order, Hereditary Rings and Dedekind Domains.

Definition 0.4.1. A ring Q is called a quotient ring if every regular element of Q is a unit (invertible), for example i) Every right (or left) Artinian ring is quotient ring, ii) Every von-Neumann regular ring is a quotient ring and iii) Any algebraic algebra over a field is a quotient algebra.

Definition 0.4.2. Given a quotient ring Q, a subring R not necessarily with 1 is called a right order in Q if each $q \in Q$ has the form $q = r s^{-1}$ for some $r, s \in R$. A left order is defined analogously.

A ring R is called order in Q if it is left and right order in Q.

The necessary and sufficient condition for a ring to be ordered in a quotient ring is given by a well known theorem of Goldie.

Theorem 0.4.3.(Goldie) A ring R is semi prime right Goldie if and only if R is a right order in an artinian ring Q.

Definition 0.4.4. Suppose that R is a right order in Q. Then a fractional right R-ideal is a submodule I of Q_R such that there exist units $a, b \in Q$ for which $aI \subseteq R$ and $bI \subseteq I$.

Examples: ½ Z is a fractional Z-ideal. In a semi prime right Goldie ring any essential right ideal I (i.e. $I \cap J \neq 0$ for any non-zero ideal J) is a fractional right ideal.

More generally, in a prime Goldie ring any ideal is a fractional ideal (since any ideal is right and left essential).

Definition 0.4.5. Suppose that R is a prime Goldie ring. A fractional R-ideal A is invertible if there exists a fractional R-ideal B with $AB = BA = R$. B is usually denoted by A^{-1}.

Definition 0.4.6. A prime Goldie ring R is called Asano order (or Asano prime ring) iff every nonzero ideal of R is invertible.

Definition 0.4.7. A ring R is called right Hereditary if any right ideal of R is projective, or equivalently every submodule of a projective module is projective.

Definition 0.4.8. A ring R is called Dedekined prime, or noncommutative Dedekind domain if R is a Hereditary Noetherian, Asano order.

§ 0.5 (PLID) Rings (Principal left ideal domain) Polynomial Identities and Tensor Products.

Definition 0.5.1. A left ideal $I \Delta_\ell R$ is called principal if $I = Ra$ for same $a \in R$

Definition 0.5.2. A ring is called (PLI) principal left ideal ring if every left ideal is principal and is called PLID if R is a domain and PLI.

If R is PLID with $\sigma(R-\{0\}) \subseteq$ Unit R then $S = R[x, \sigma]$ is a PLID.

Definition 0.5.3. A ring (an algebra) R with center (over a commutative ring) C is called to satisfy a polynomial identity (PI) of degree n if there is a polynomial $f(x_1, x_2, \ldots, x_n) \in C[x_1, x_2, \ldots, x_n]$; the free algebra over C with n-variables $x_1, \ldots, x_n$, such that $f(a_1, \ldots, a_n) = 0$ for all $a_1, a_2, \ldots, a_n \in R$.

Of course, any commutative ring satisfies a P.I since $f(r_i, r_j) = r_i r_j - r_j r_i = 0$ for r_i, $r_j \in R$.

Definition 0.5.4. Let A, B be F-algebras. Then the tensor product of the algebras A, B is

$\{\Sigma (a_i \otimes b_i) \mid a_i \in A, b_i \in B\}$ such that

i) $((\Sigma_i a_i) \otimes b_j) = \Sigma_i (a_i \otimes b_j)$ and $(a_i \otimes (\Sigma b_j)) = \Sigma_j (a_i \otimes b_j)$

ii) $\alpha a_i \otimes b_j = \alpha (a_i \otimes b_j) = (a_i \otimes \alpha b_j)$ and

iii) $(a_i \otimes b_j)(a_i' \otimes b_j') = a_i a_i' \otimes b_j b_j'$. It is well know that

1- if $\dim_F A = n$ and $\dim_F B = m$, then $\dim A \otimes_F B = mn$.

2- if A is an F-algebra and $M_n(F) = F_n$, (where F_n is the n X n matrices over F), then $A \otimes_F F_n \approx A_n$.

3- if A is an F-algebra and F[x], (F[[x]]) is the ring of polynomials (ring of power series), then

$$A \otimes_F F[x] \approx A[x], (A \otimes F[[x]] \approx A[[x]]).$$

§ 0.6 Polycyclic - by - Finite Groups.

Definition 0.6.1. The infinite Dihedral group $D_\infty = \langle a,b \mid b^{-1}ab = a^{-1} \text{ and } b^2 = 1 \rangle$.

D_∞ contains infinite normal subgroups of finite index e.g. the subgroups $\langle a \rangle_\infty$, $\langle a^2, b \rangle$ and infinite number of subgroups of order 2 namely the subgroups $M_i = \langle a^i b \rangle_2$ for all i. Non of subgroups M_i is normal.

Definition 0.6.2. A subnormal series of subgroups of G is a chain $G = G_m \triangleright G_{m-1} \triangleright \ldots \triangleright G_o = \{1\}$, i.e. each $G_{i-1} \triangleleft G_i$

Definition 0.6.3. A group G is polycyclic (poly-infinite cyclic) if G has a subnormal series with each factor G_i/G_{i-1} cyclic (infinite cyclic).

Definition 0.6.4. G is polycyclic - by - finite, if G has a polycyclic group of finite index.

Thoerem 0.6.5. G is polycyclic - by - finite, if G has a subnormal series whose factors are finite or cyclic.

Moreover, any polycyclic - by - finite group has a characteristic poly {infinite cyclic} subgroup H of finite index.

Definition 0.6.6. G is solvable group if there is a subnormal series with each factor abelian.

Definition 0.6.7. G is supersolvable group if it has a normal series whose factors are cyclic.

Definition 0.6.8. G is nilpotent if it has a central series, that is a normal series $(1) = G_o \leq G_1 \leq \ldots \leq G_n = G$ such that $G_i/G_{i-1} \subset \text{center } (G/G_{i-1})$ for all i

Thus nilpotent groups and supersolvable groups are polycyclic and every polycyclic group is solvable.

The infinite dihedral group D_∞ is an easy but enlightening example of nonnilpotent polycyclic group.

Definition 0.6.9. The Hirsch number h(G) of a polycyclic - by - finite group G is the number of infinite cyclic factors in a subnormal series. h(G) is unique for a given group G.

§ 0.7 Finite Fields, Absolute Fields and More about Modules.

Definition 0.7.1. A field K is called ABSOLUTE iff K is an algebraic over a finite field which is equivalent to $k^{n(k)} = k$, for every $k \in K$.

The next proposition though it can be deduced by routine calculations but has many important applications.

proposition 0.7.2. if K is a finite field, $|K| = p^n$, where p is a prime number and n is a positive integer. Let L is a finite extension of K, then $L = p^m$, where $m = ns$ for some positive integers s.

K^* and L^* are cyclic groups generated by η, ξ the $(p^n - 1)^{th}$ and $(p^m - 1)^{th}$ roots of unity respectively and $\xi^t = \eta$, where $t = \sum_{i=1}^{s} p^{sn-in}$ (e.g. if $m = 2n$, then $t = p^n+1$

The group G(L;K): automorphism group of L which keeps K fixed has an order which equals $\frac{m}{n} = s$.

Definition 0.7.3. A module M_R is called compressible if for all nonzero submodules N of M there exists a monomorphism $M \rightarrow N$.

In particular every uniform ideal of a prime Goldie ring is compressible.

Definition 0.7.4. A module M_R is called stably free of rank t iff $M \oplus R^S = R^{S+t}$.

It can be easily deduced that every finitely generated free module is stably free and that every stably free module is finitely generated projective.

Definition 0.7.5. A ring R is called right regular ring (r-regular) iff every finitely generated R-module has finite projective dimension, (or iff every cyclic R-module has finite projective dimension).