Summary of Ph.D

Since the introductory paper of Lotfi A. Zadeh [60], significant strides have been made in the mathematics of fuzzy subsets. Much of this mathematical development has been increasingly categorical in nature, particularly in the areas of Logic and Topology, as well as in the overlap of these two areas.

With respect to Topology, fuzzy set theory has been applied much since Chang introduced his notion of fuzzy topology in [22]. Several other authors continued the investigation of such spaces such as Eklund and Gähler [25], Erceg [26], Gähler et al. [28, 29, 30, 31, 32], Goguen [34], Gottwald [35], Kaleva and Seikkala [42], Kandil et al. [43, 44], Katsaras and Petalas [47], Lowen [49, 50, 51, 52] and Wong [58]. In Goguen's paper [34], he considered as values of the fuzzy sets not only elements of the closed unit interval but elements of some more general lattice L. A fuzzification of the classical proximity [53] were given by Artico and Moresco [3, 4], Gähler et al. [31], Bayoumi [19] and by Katsaras [45]. Also, the uniform spaces [57] have been developed into the fuzzy case in [5, 6, 17, 20, 32, 41]. The metric space has been generalized to the fuzzy case in [26, 29, 42]. Also much interest has been shown to generalize separation axioms to fuzzy separation axioms [7, 8, 9, 10, 11, 12, 13, 43].

The notion of topological groups [37, 40] was one of the items that many authors tried to introduce and study in the fuzzy case such as in [2, 27]. In [27], Foster brought together the structure of fuzzy topological space given by Lowen [49] and the structure of fuzzy group he defined by himself. The fuzzy topological group (G, τ) in [2] is defined by Ahsanullah as an ordinary group G equipped with a fuzzy topology τ on G such that the binary operation and the unary operation of the inverse are fuzzy continuous with respect to τ.

The main notion of this thesis is the notion of fuzzy topological group. We study this fuzzy topological group and investigate its relation with the fuzzy separation axioms (specially the fuzzy Tychonoff axiom), the fuzzy uniform spaces, the fuzzy proximity spaces and the fuzzy metric spaces. The fuzzy separation axioms and the fuzzy uniform spaces, which we use here, are defined using the notion of fuzzy filter. This notion of fuzzy filter has been introduced in [25, 29]. By means of this fuzzy filter a point-based approach to fuzzy topology related to usual points has been developed [29, 30]. For agreement reasons with the fuzzy filter, we preferred the notion of fuzzy topological group defined by Ahsanullah in [2].

In [38] much progress was made in standardizing terminology in the overall discipline of fuzzy topology. In particular, the notation presented in [39, 56], reached

after much negotiation and final agreement with all the authors of [38, 48], is that the kind of topologies being defined in Chang and Goguen are not fuzzy topologies, but rather topologies of fuzzy subsets. If a lattice L of membership values has been chosen, then the corresponding Chang-Goguen spaces are "L-topological spaces" and the mappings from L^X to L which satisfy certain properties are called "L-fuzzy topologies" and spaces with such topologies are "L-fuzzy topological spaces". To standardize the terminology and notation of topology, it is important to look at [39, 56] and make our notation consistent with them. In particular, the phrases "fuzzy topological group", "fuzzy separation axiom", "fuzzy uniform space", "fuzzy proximity space", "fuzzy metric space", "fuzzy filter", "fuzzy set", "fuzzy real number" and "fuzzy continuous" should be termed "L-topological group", "L-separation axiom", "L-uniform space", "L-proximity space", "L-metric space", "L-filter", "L-set", "L-real number" and "L-continuous", respectively.

In [2, 14, 15, 16, 18], many results on the L-topological groups are studied. We show in this thesis that any L-topological group is uniformizable and that any L-topological group (separated) is pseudo-metrizable (metrizable). We also introduce the notions of complete L-uniform spaces and complete L-topological groups and hence we construct the completion of the L-uniform spaces and of the L-topological groups. Some examples for the L-topological groups are given.

This L-topological group provides the expected relations with the L-separation axioms. To show this relation, we introduce and study, in this thesis, new kinds of L-separation axioms for the L-topological spaces depending on usual points, ordinary subsets and the L-real numbers [28]. These axioms are good extensions, in sense of Lowen [51], of the related separation axioms in the classical case [23, 24]. We denote by GT_i for these axioms and by a GT_i-space for the L-topological space which provides the axiom GT_i. The GT_i-spaces, $i = 0, 1, 2, 3, 4$ have been introduced and studied by the authors in [7, 8, 9, 10]. We introduce and study here the case of the GT_i-spaces, $i = 2\frac{1}{2}, 3\frac{1}{2}, 5, 6$. The L-neighborhood filters at a point and at an ordinary set will be used to define the axioms $GT_{2\frac{1}{2}}, GT_5$. Using the notions of the L-real numbers and the L-unit interval [28], we define the $GT_{3\frac{1}{2}}$-spaces and the GT_6-spaces. Many of the expected relations between these axioms are fulfilled. We also introduced a number of examples to illustrate and assure our results. The initial and the final structures [50] of these GT_i-spaces also are GT_i-spaces, $i = 2\frac{1}{2}, 3\frac{1}{2}, 5, 6$. A comparison study will be done between the $GT_{3\frac{1}{2}}$-spaces and other notions of fuzzy $T_{3\frac{1}{2}}$-spaces [41, 43, 46]. Moreover, the L-proximity spaces [31, 45], the L-uniform spaces [32] and the G-compact spaces [30] have good relations with the $GT_{3\frac{1}{2}}$-spaces. It is shown that the L-topology of an L-topological group (resp. separated L-topological group) is completely regular (resp. $GT_{3\frac{1}{2}}$).

For the lattice theory we refer to [21, 33] and for the category theory we refer to [1, 36, 59].

The thesis consists of six chapters and is organized as follows:

Each chapter begins with an introduction.

Chapter 1 contains the motivations, ideas, definitions and results about the notions which we shall use throughout the thesis. We recall the notions of L-sets, L-topological spaces, L-filters, L-neighborhood filters, L-real numbers, L-metric spaces, L-proximity spaces, L-uniform spaces and G-compact spaces.

Chapter 2 is devoted to introduce the notion of completely regular spaces and $GT_{3\frac{1}{2}}$-spaces and study all possible relations with the L-proximity spaces, the L-uniform spaces and the G-compact spaces. The initial and the final cases also are studied. A study on the relation between the $GT_{3\frac{1}{2}}$-spaces and other fuzzy $T_{3\frac{1}{2}}$-spaces are given.

This chapter consists of seven sections. **Section 2.1** is an introduction. In **Section 2.2**, we introduce the notion of completely regular spaces and $GT_{3\frac{1}{2}}$-spaces. **Section 2.3** is given to study the initial and the final cases of the $GT_{3\frac{1}{2}}$-spaces. **Section 2.4** is given to show that our completely regular spaces are more general than the completely regular spaces defined by Hutton in [41], by Katsaras in [46] and by Kandil and El-Shafee in [43]. In **Section 2.5**, we study the relation between the $GT_{3\frac{1}{2}}$-spaces and the L-proximity spaces defined by Katsaras in [45]. **Section 2.6** is devoted to show the relation between the $GT_{3\frac{1}{2}}$-spaces and the L-uniform spaces defined in [32]. **Section 2.7** is devoted to study the relation of the $GT_{3\frac{1}{2}}$-spaces with the L-compact spaces in sense of Gähler [30] which is denoted by G-compact spaces.

> Let us here mention that the most of results in Chapter 2 are published in [11, 12].

We have introduced the GT_i-spaces, $i = 0, 1, 2, 3, 4$ in [7, 8, 9, 10] and the $GT_{3\frac{1}{2}}$-spaces in Chapter 2 of this thesis and now to complete our generalization for the separation axioms, it remains to introduce the GT_i-spaces, $i = 2\frac{1}{2}, 5, 6$.

Chapter 3 is devoted to introduce the notions of $GT_{2\frac{1}{2}}$-spaces, GT_5-spaces and GT_6-spaces. The initial and the final cases of $GT_{2\frac{1}{2}}$-spaces, GT_5-spaces and GT_6-spaces also are studied.

This chapter consists of seven sections. **Section 3.1** is an introduction. In **Section 3.2**, we introduce the notion of $GT_{2\frac{1}{2}}$-spaces. **Section 3.3** is given to study the initial and the final cases of the $GT_{2\frac{1}{2}}$-spaces. **Section 3.4** is devoted to introduce the notions of completely normal spaces and GT_5-spaces. In **Section 3.5**, we study the initial and the final cases of the GT_5-spaces. In **Section 3.6** and **Section 3.7**, we introduce the notion of GT_6-spaces and study the initial and the final cases of the GT_6-spaces, respectively.

All results in Chapter 3 are submitted for publication in [13].

In **Chapter 4**, the L-neighborhood filter at the identity element of the L-topological group (G, τ) plays an important role. The L-neighborhood filter at the identity element of the L-topological group (G, τ) corresponds a family of prefilters on G [29]. We show that for each L-topological group (G, τ), there are unique left and right invariant L-uniform structures on G compatible with τ. Moreover, we study in this chapter some relations between the L-topological groups and the L-separation axioms GT_i which we had introduced in [8, 9, 11].

This chapter consists of four sections. **Section 4.1** is an introduction. In **Section 4.2**, we give the definitions and notations of L-topological groups. **Section 4.3** is devoted to show that using this family of prefilters, we constructed in this chapter, we can introduce a unique left invariant L-uniform structure $\mathcal{U}^l$ and a unique right invariant L-uniform structure $\mathcal{U}^r$ on G. These L-uniform structures $\mathcal{U}^l$ and $\mathcal{U}^r$ are compatible with τ, which means that the L-topological group (G, τ) is uniformizable. The L-uniform structures $\mathcal{U}^l$ and $\mathcal{U}^r$ are defined as L-filters on the cartesian product $G \times G$. We also show that for any group G and any family of prefilters fulfills certain conditions, we can define the left and the right L-uniform structures $\mathcal{U}^l$ and $\mathcal{U}^r$ on G such that $\tau_{\mathcal{U}^l} = \tau_{\mathcal{U}^r}$ is an L-topology τ on G for which the pair (G, τ) is an L-topological group. Moreover, this family of prefilters is exactly the family of prefilters which corresponds the L-neighborhood filter at the identity element of the L-topological group (G, τ). In **Section 4.4**, we show that the L-topology τ of an L-topological group (G, τ) is completely regular in our sense [9] and that the L-topological group (G, τ) is separated if and only if the L-topology τ is GT_0 (resp. GT_1, GT_2, $GT_{3\frac{1}{2}}$) if and only if the left L-uniform structure $\mathcal{U}^l$ (resp. the right L-uniform structure $\mathcal{U}^r$) is separated.

Most of results in Chapter 4 are accepted for publication in [14].

In **Chapter 5**, using the L-pseudo-metric (L-metric) in sense of [42], we introduce and study the metrizability of an L-topological group. This L-metric on a set G is defined as a mapping of the cartesian product $G \times G$ to the set of all L-real numbers in sense of [28].

This chapter consists of three sections. **Section 5.1** is an introduction. **Section 5.2** introduces and shows some results on the L-metric spaces and the L-uniform spaces, which are needed to show the metrizability of L-topological groups. The first countable L-topological spaces are defined. In **Section 5.3**, we show that the L-pseudo-metric (L-metric), in sense of [42], induces the L-topology of an L-topological group (separated), that is, any L-topological group (separated) is pseudo-metrizable (metrizable).

These results in Chapter 5 are accepted for publication in [15].

In **Chapter 6**, the notions of complete L-uniform spaces and complete L-topological groups are introduced and hence the completion of an L-uniform space and the completion of an L-topological group are investigated. We defined here the product L-filter of two L-filters in general and we also defined, in an L-uniform space $(X,\mathcal{U})$, the notion of $\mathcal{U}$-cauchy filter.

This chapter consists of four sections. **Section 6.1** is an introduction. In **Section 6.2**, we introduced, in the L-uniform space $(X,\mathcal{U})$ in sense of [32], the $\mathcal{U}$-cauchy filters. It is proved that every convergent L-filter in an L-uniform space $(X,\mathcal{U})$ is a $\mathcal{U}$-cauchy filter. The product of two L-filters is illustrated and also some results are proved. **Section 6.3** is devoted to introduce the complete L-uniform spaces. An L-uniform space $(X,\mathcal{U})$ is called complete if every $\mathcal{U}$-cauchy filter $\mathcal{M}$ on X is convergent. Moreover, an L-uniform space $(Y,\mathcal{U}^*)$ is called a completion of the L-uniform space $(X,\mathcal{U})$ if it is a reduced extension of $(X,\mathcal{U})$ and $\mathcal{U}^*$ is complete. Hence, we construct the completion of the L-uniform spaces. **Section 6.4** is devoted to extending an L-topological group into a complete L-topological group, and so find a way to get the completion of L-topological group. The completion of the L-uniform spaces will be used to get the completion of the L-topological groups.

Here, we mention that the most of results in Chapter 6 are submitted for publication in [16].