Chapter 2

$GT_{3\frac{1}{2}}$-Spaces and Completely Regular Spaces

2.1 Introduction

There is a notion of L-real numbers introduced by S. Gähler and W. Gähler in [28], and is defined as a convex, normal, compactly supported and upper semi-continuous L-subsets of the set of real numbers $\mathbf{R}$. The set of all L-real numbers is called *L-real line* and is denoted by $\mathbf{R}_L$, where L is a complete chain. In this chapter, using the space $(I_L, \Im)$, where $I = [0,1]$ is the closed unit interval and $\Im$ is the L-topology on I_L, a notion of completely regular L-topological spaces is introduced and studied.

In Section 2.2, this completely regular L-topological space is defined, as in case of GT_i-spaces; $i = 0, 1, 2, 3, 4$, using the ordinary points and usual subsets. The L-topological space which is GT_1 and completely regular in our sense will be denoted here by $GT_{3\frac{1}{2}}$-space (or L-Tychonoff space) and the category of all $GT_{3\frac{1}{2}}$-spaces will be denoted by L-**Tych**. For these $GT_{3\frac{1}{2}}$-spaces, the Urysohn Lemma is proved and hence it is shown that each GT_4-space is a $GT_{3\frac{1}{2}}$-space. Moreover, each $GT_{3\frac{1}{2}}$-space is a GT_3-space. For each case a counter example will be given. It also is shown that the $GT_{3\frac{1}{2}}$-space is an extension with respect to the functor ω, defined by Lowen in [51], from the category **Tych** of $T_{3\frac{1}{2}}$-spaces to the category L-**Tych**.

We showed in Section 2.3 that the category L-**Tych** is topological over the category **Set** of sets [1]. This means that the initial and the final L-topological spaces of a family of $GT_{3\frac{1}{2}}$-spaces also are $GT_{3\frac{1}{2}}$-spaces. As special initial and final L-topological spaces, the subspace, the product space, the quotient space and the sum space of $GT_{3\frac{1}{2}}$-spaces are $GT_{3\frac{1}{2}}$-spaces.

There are several notions of completely regular L-topological spaces such as the notions defined by Hutton in [41], by Katsaras in [46] and by Kandil and El-Shafee in [43]. In Section 2.4, it is shown that our notion of completely regular L-topological spaces is more general than these notions [41, 43, 46]. Counter examples are given to show these generalizations.

In Section 2.5, we shall study the relation between the $GT_{3\frac{1}{2}}$-spaces and the L-proximity spaces defined by Katsaras in [45]. Using Urysohn's Lemma, which we had established in Section 2.2 and other results which are proved here, we show many results joining the completely regular L-topology in our sense and the L-proximity in sense of Katsaras. We show that the L-topology associated with any L-proximity is completely regular in our sense. Moreover, we show that every completely regular stratified L-topology is compatible with an L-proximity.

Section 2.6 is devoted to study the relation between the $GT_{3\frac{1}{2}}$-spaces and the L-uniform spaces defined by Gähler and the first author and others in [32]. In these L-uniform spaces $(X, \mathcal{U})$, the L-uniform structures $\mathcal{U}$ are defined, in a similar way to the usual case, as L-filters on $X \times X$. We had established some results similar to what we had introduced for the L-proximities in Section 2.5. We show that the L-topology associated with any L-uniform structure is completely regular in our sense, and that every completely regular stratified L-topology is compatible with an L-uniform structure, that is, every completely regular stratified L-topology is uniformizable.

In the last section, Section 2.7, we investigate the relation of the $GT_{3\frac{1}{2}}$-spaces with the L-compact spaces defined by Gähler in [30], which is called G-compact spaces. We show also here some results joining the $GT_{3\frac{1}{2}}$-spaces and the G-compact spaces. We show that the L-unit interval $(I_L, \Im)$ and that the L-cube, defined as a product of L-unit intervals are G-compact GT_2-spaces and consequently GT_4-spaces, and hence they are $GT_{3\frac{1}{2}}$-spaces. We also show that a G-compact space is a GT_2-space if and only if it is a $GT_{3\frac{1}{2}}$-space. If τ_1 and τ_2 are L-topologies on a set X with τ_1 is finer than τ_2, and (X, τ_1) is a G-compact space and (X, τ_2) is a $GT_{3\frac{1}{2}}$-space, then we prove that τ_1 is equivalent to τ_2. Moreover, we show that an L-topological space (X, τ) is a $GT_{3\frac{1}{2}}$-space if and only if it is homeomorphic to a subspace of an L-cube if and only if it is homeomorphic to a subspace of a G-compact GT_2-space if and only if it is homeomorphic to a subspace of a GT_4-space.

2.2 $GT_{3\frac{1}{2}}$-spaces

Now, we shall introduce our notion of completely regular spaces in the fuzzy case.

Definition 2.2.1 An L-topological space (X, τ) is said to be *completely regular* if for all $x \in X$, $F \in P(X)$ with $F \in \tau'$ and $x \notin F$, there exists an L-continuous mapping $f : (X, \tau) \to (I_L, \Im)$ such that $f(x) = \overline{1}$ and $f(y) = \overline{0}$ for all $y \in F$.

Definition 2.2.2 An L-topological space (X, τ) is called a $GT_{3\frac{1}{2}}$-*space* (or an *L-Tychonoff space*) if it is GT_1 and completely regular.

From that $\bigwedge_{s<t} f(z)(s) \geq \bigvee_{r>t} f(z)(r)$ for all $z \in X$ in general, we get for all $z \in X$ that:

$$\begin{aligned}(h \wedge k)(z) &= ((R_{\frac{1}{2}} \circ f) \wedge (R^{\frac{1}{2}} \circ f))(z)\\ &= \bigvee_{\alpha>\frac{1}{2}} f(z)(\alpha) \wedge (\bigvee_{\alpha\geq\frac{1}{2}} f(z)(\alpha))'\\ &\leq \bigwedge_{\alpha<\frac{1}{2}} f(z)(\alpha) \wedge \bigwedge_{\alpha\geq\frac{1}{2}} f(z)(\alpha)'\\ &< 1.\end{aligned}$$

Hence, $\sup(h \wedge k) < \mathcal{N}(x)(k) \wedge \bigwedge_{y\in F} \mathcal{N}(y)(h)$ and therefore (X, τ) is a regular space and consequently it is a GT_3-space. □

In this example we introduce a GT_3-space which is not a $GT_{3\frac{1}{2}}$-space.

Example 2.2.2 Let $X = \{x, y\}$ with $x \neq y$ and let $\tau = \{\overline{0}, \overline{1}, y_{\frac{1}{2}}, y_1, x_{\frac{3}{4}} \vee y_{\frac{1}{2}}, x_{\frac{3}{4}} \vee y_1\}$. Then $\tau' = \{\overline{0}, \overline{1}, x_{\frac{1}{4}}, x_1, x_{\frac{1}{4}} \vee y_{\frac{1}{2}}, x_1 \vee y_{\frac{1}{2}}\}$ and there is only the case of $y \notin \{x\} \in \tau'$ to be studied. Since $f = x_{\frac{3}{4}} \vee y_{\frac{1}{2}}$ and $g = y_1$ in L^X implies

$$\mathcal{N}(x)(f) \wedge \mathcal{N}(y)(g) = \text{int}_\tau f(x) \wedge \text{int}_\tau g(y) = \frac{3}{4} > \frac{1}{2} = \sup(f \wedge g),$$

then $\mathcal{N}(x) \wedge \mathcal{N}(y)$ does not exist and hence (X, τ) is a regular space and it also is a GT_1-space. Thus (X, τ) is a GT_3-space.

Since in case of $y \notin \{x\} \in \tau'$ we get that any mapping $f : (X, \tau) \to (I_L, \Im)$ such that $f(y) = \overline{1}$ and $f(x) = \overline{0}$ is not L-continuous, then (X, τ) is not completely regular and thus it is not a $GT_{3\frac{1}{2}}$-space.

Let $X \neq \emptyset$ be an arbitrary set. By an *L-function family* Φ on X, we mean the set of all L-real functions $f : X \to I_L$.

Let f and g be L-sets in X. Then a function $h : X \to I_L$ is said to *separate* f and g if $\overline{0} \leq h(x) \leq \overline{1}$ for all $x \in X$, $x_1 \leq f$ implies $h(x) = \overline{1}$ and $y_1 \leq g$ implies $h(y) = \overline{0}$. Moreover, if Φ is an L-function family on X, then the sets $f, g \in L^X$ are called *Φ-separated* or *Φ-separable* if there exists a function $h \in \Phi$ separating them.

Let $(\ll_n)$ be a sequence of L-topogenous structures on X and $(\prec_n)$ a sequence of L-topogenous structures on I_L. Then an L-real function $f : X \to I_L$ is called *associated with* the sequence $(\ll_n)$ if for all $g, h \in L^{I_L}$, $g \prec_n h$ implies $(g \circ f) \ll_{n+1} (h \circ f)$ for every positive integer n.

Remark 2.2.1 Consider $(\ll_n)$ and $(\prec_n)$ are two sequences of two complementarily symmetric L-topogenous structures $\ll$ and $\prec$ on X and I_L, respectively. Let δ and

δ^* be the L-proximities on X and I_L identified with $\ll$ and $\prec$ by (1.6), respectively. Then for a function $f : X \to I_L$ associated with the sequence $\ll$, we get from (1.6) that $g\,\overline{\delta^*}\,h$ implies $(g \circ f)\,\overline{\delta}\,(h \circ f)$ for all $g, h \in L^{I_L}$, which means that f is L-proximally continuous.

Here, to prove Urysohn's Lemma for our notion of $GT_{3\frac{1}{2}}$-spaces, we need the following results.

In the proof of the following lemma we use the way of Császár [24].

Lemma 2.2.1 *Suppose that $\ll_n$ $(n = 0, 1, 2, \ldots)$ are complementarily symmetric L-topogenous structures on a set X. If $F, G \in P(X)$ and $\chi_F \ll_0 \chi_G$, then there exists a function $f : X \to I_L$ associated with the sequence $(\ll_n)$ for which $f(x) = \overline{0}$ for all $x \in F$ and $f(y) = \overline{1}$ for all $y \in G'$.*

Proof. Since $(\ll_n)$ is a sequence of binary relations in the crisp case and fulfill the conditions of being complementarily symmetric L-topogenous structures, then we can deduce that there is a recursion process in the crisp case similar to that in the usual case in [24] by defining the order relation $\ll_m$ for $m \in R$ where R denotes the set of all non-negative dyadic rational numbers $(m = \frac{p}{2^n}; p = 0, \ldots, 2^n, n = 0, 1, \ldots)$. With this relation, the sets $A(m)$ can be associated such that

(1) $A(0) = F$, $A(1) = G$;

(2) $A(\frac{p}{2^n}) \ll_m A(\frac{p+1}{2^n})$, $A(r) = X$ for all $r > 1$; $p = 0, \ldots, 2^n, n = 0, 1, \ldots$.

From the properties of $\ll_n$ we get $A(r) \subseteq A(s)$ for all $r, s \in R$, $r < s$.

Define the L-real function $f : X \to I_L$ by $f(x) = \bigwedge_{r \in R} \{\overline{r} \mid x \in A(r)\}$, then $f(x) = \overline{0}$ for all $x \in F$ and $f(y) = \overline{1}$ for all $y \in G'$.

Now, as in the usual case, f itself is an associated function with the sequence $(\ll_n)$ and also f separates the sets χ_F and $\chi_{G'}$. Hence the proof is complete. □

From (1.6), Lemma 2.2.1 and Remark 2.2.1 we can easily deduce the following.

Proposition 2.2.2 *Let F and G be subsets of X with $\chi_F\,\overline{\delta}\,\chi_G$ in the L-proximity space (X, δ) and let Φ be the family of those L-proximally continuous functions h of (X, δ) into the L-proximity space (I_L, δ^*) for which $x \in X$ implies $\overline{0} \leq h(x) \leq \overline{1}$. Then χ_F and χ_G are Φ-separable.*

Proof. Let $\ll$ be the complementarily symmetric L-topogenous structure identified with δ. From (1.6), $\chi_F\,\overline{\delta}\,\chi_G$ implies that $\chi_F \ll \chi'_G$ and since $h \in \Phi$ is L-proximally continuous, then h, by means of Remark 2.2.1, is associated with $\ll$.